Bearing Capacity of Strip Footings on Spatially Random Soils Using Sparse Polynomial Chaos Expansion

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ABSTRACT

In this paper, a numerical model that considers the spatial variability of the soil properties is presented to compute the probability density function (PDF) of the ultimate bearing capacity of strip footings resting on spatially random soils. An efficient uncertainty propagation methodology was employed in this paper. This methodology makes use of a sparse polynomial chaos expansion (SPCE) for the system response. The probabilistic numerical results have shown that (i) the autocorrelation distance has significant effect on the PDF of the bearing capacity, (ii) the probabilistic mean value of the bearing capacity presents a minimum for a given value of the ratio of the horizontal and vertical autocorrelation distances, (iii) the coefficient of variation of the cohesion parameter has a more significant influence than that of the friction angle on the variability of the response and finally (iv) the negative correlation between the soil shear strength parameters decreases the response variability.

INTRODUCTION

In this paper, the probability density function (PDF) of the ultimate bearing capacity of a shallow strip foundation resting on a spatially varying soil is computed using the sparse polynomial chaos expansion (SPCE) methodology. The deterministic model is based on numerical simulations using FLAC^{3D} software. The soil cohesion and friction angle are considered as two anisotropic non-Gaussian cross-correlated random fields. The methodology proposed by Vořechovsky (2008) is used to generate the two random fields. The adaptive algorithm suggested by Blatman and Sudret (2010) to build up a SPCE is used to obtain a meta-model (i.e. an approximate analytical expression) of the ultimate bearing capacity. Finally, this meta-model is employed to perform the probabilistic analysis using Monte Carlo simulation technique. The paper is organized as follows: The first two sections aim at presenting the method used to generate the two random fields and the SPCE methodology employed to determine the analytical expression of the system response. They are followed by a presentation and a discussion of the probabilistic numerical results.

METHOD OF GENERATION OF ANISOTROPIC CROSS-CORRELATED NON-GAUSSIAN RANDOM FIELDS

Let us consider two anisotropic non-Gaussian cross-correlated random fields $Z_i^{NG}(x, y)\ (i = c, \varphi)$ described by: (i) constant means and standard deviations (μ_i, σ_i; $i = c, \varphi$), (ii) non-Gaussian marginal cumulative distribution functions G_i ($i = c, \varphi$), (iii) a target cross-correlation matrix C^{NG} and (iv) a common square exponential autocorrelation function ρ_Z^{NG} [(x, y), (x', y')] which gives the values of the correlation function between two arbitrary points (x, y) and (x', y'). This autocorrelation function is given as follows:

$$\rho_z^{NG}[(x, y), (x', y')] = \exp\left(-\left(\frac{x - x'}{a_x}\right)^2 - \left(\frac{y - y'}{a_y}\right)^2 \right) \tag{1}$$

where a_x and a_y are the autocorrelation distances along x and y respectively. The Expansion Optimal Linear Estimation method (EOLE) and its extension to cover the case of correlated non-Gaussian random fields are used herein to generate the two random fields of c and φ. Notice that EOLE was first proposed by Li and Der kiureghian (1993) for the case of uncorrelated Gaussian fields, and then extended by Vořechovsky (2008) to cover the case of correlated non-Gaussian fields. In this method, one should first define a stochastic grid composed of q grid points (or nodes) $\left\{(x_1, y_1), ..., (x_q, y_q)\right\}$ for which the values of the field are assembled in a vector $\chi = \left\{Z(x_1, y_1), ..., Z(x_q, y_q)\right\}$. Then, one should determine the correlation matrix for which each element $\left(\Sigma_{\chi;\chi}\right)_{i,j}^{NG}$ is calculated as follows:

$$\left(\Sigma_{\chi;\chi}\right)_{i,j}^{NG} = \rho_z^{NG}\left[(x_i, y_i), (x_j, y_j)\right] \tag{2}$$

The common non-Gaussian autocorrelation matrix $\Sigma_{\chi;\chi}^{NG}$ and the target non-Gaussian cross-correlation matrix C^{NG} should be transformed into the Gaussian space using Nataf model [Nataf (1962)] since the discretization of the random fields using EOLE is done in the Gaussian space. As a result, one obtains two Gaussian autocorrelation matrices $\Sigma_{\chi;\chi}^{c}$ and $\Sigma_{\chi;\chi}^{\varphi}$, and a Gaussian cross-correlation matrix C that can be used to discretize the two random fields as follows:

$$\tilde{Z}_i(x, y) = \mu_i + \sigma_i \sum_{j=1}^{N} \frac{\kappa_{i,j}^{D}}{\sqrt{\lambda_j^i}} .\phi_j^i .\Sigma_{Z(x,y);\chi}^{i} \qquad i = c, \varphi \tag{3}$$

where $(\kappa_{i,j}^{D}; i = c, \varphi)$ are two cross-correlated blocks of independent standard normal random variables obtained using the Gaussian cross-correlation matrix C between the two fields [for more details, the reader may refer to Vořechovsky (2008)]. Notice finally that $(\lambda_j^i, \phi_j^i; i = c, \varphi)$ in equation (3) are the eigenvalues and eigenvectors of the two Gaussian autocorrelation matrices $\Sigma_{\chi;\chi}^{c}$ and $\Sigma_{\chi;\chi}^{\varphi}$ respectively, and $\Sigma_{Z(x,y);\chi}$ is the correlation vector between the random vector χ and the value of the field at an arbitrary point (x, y). Once the two Gaussian random fields are obtained, they should be transformed into the non-Gaussian space by applying the following formula:

$$\tilde{Z}_i^{NG}(x, y) = G_i^{-1}\left\{\Phi\left[\tilde{Z}_i(x, y)\right]\right\} \qquad i = c, \varphi \tag{4}$$

where $\Phi(.)$ is the standard normal cumulative density function.

SPARSE POLYNOMIAL CHAOS EXPANSION (SPCE) METHODOLOGY

The polynomial chaos expansion (PCE) methodology aims at replacing a complex deterministic model (i.e. finite element/finite difference numerical model) whose input parameters are modeled by random variables by a meta-model which allows one to calculate the system response using an approximate simple analytical equation [Huang et al. (2009), Blatman and Sudret (2010)]. The coefficients of the PCE are computed herein using a regression approach.

For a deterministic numerical model with M input uncertain parameters, the uncertain parameters should first be represented by independent standard random variables $\{\xi_i\}_{i=1,\ldots,M}$ gathered in a random vector ξ. The random response Γ of our mechanical model can then be expressed by a PCE of order p fixed by the user as follows:

$$\Gamma_{PCE}(\xi) = \sum_{\beta=0}^{\infty} a_\beta \Psi_\beta(\xi) \cong \sum_{\beta=0}^{P-1} a_\beta \Psi_\beta(\xi) \tag{5}$$

Where P is the number of terms retained in the truncation scheme, a_β are the unknown PCE coefficients to be computed and Ψ_β are multivariate (or multidimensional) Hermite polynomials which are orthonormal with respect to the joint probability distribution function of the standard normal random vector ξ. These multivariate polynomials can be obtained from the product of the one-dimensional Hermite polynomials [Huang et al. (2009)] as follows:

$$\Psi_\beta = \prod_{i=1}^{M} H_{\alpha_i}(\xi) \tag{6}$$

where $H_{\alpha_i}(.)$ is the α_i-th one-dimensional Hermite polynomial and α_i are a sequence of M non-negative integers $\{\alpha_1, \ldots, \alpha_M\}$. In practice, one should truncate the PCE representation by retaining only the multivariate polynomials of degree less than or equal to the PCE order p. For this reason, a classical truncation scheme based on the determination of the first order norm is generally adopted in the literature. This first order norm is defined as follows: $\|\alpha\|_1 = \sum_{i=1}^{M} \alpha_i$. The classical truncation scheme suggests that the first order norm should be less than or equal to the order p of the PCE. Using this method of truncation, the number P of the unknown PCE coefficients is given by $P = \frac{(M+p)!}{M!\,p!}$. Thus, the number P of the PCE coefficients increases dramatically with the number M of the random variables and the order p of the PCE. To overcome such a problem, Blatman and Sudret (2010) have shown that the number of significant terms in a PCE is relatively small since the multidimensional polynomials Ψ_β corresponding to high-order interaction (i.e. those resulting from the multiplication of the H_{α_i} with increasing α_i values) are associated with very small values for the coefficients a_β. Thus, a truncation strategy based on this observation was developed by Blatman and Sudret (2010) in which the multidimensional polynomials Ψ_β corresponding to high-order interaction were penalized. This was performed by considering the hyperbolic truncation scheme that considers the q-norm instead of the first order norm. The q-norm is given by $\|\alpha\|_q = \left(\sum_{i=1}^{M} \alpha_i^q\right)^{1/q}$ where q is a coefficient (0<q<1). The hyperbolic truncation scheme suggests that the q-norm should be less than or equal to the order p of the PCE. In this formula, q can be chosen arbitrarily. Blatman and Sudret (2010) have shown that sufficient accuracy is obtained for $q \geq 0.5$. Below this value, we may risk to reject some significant terms.

The proposed methodology leads to a sparse polynomial chaos expansion SPCE that contains a small number of unknown coefficients which can be calculated from a reduced number of calls of the deterministic model. This is of particular interest in the present case of random fields which involve a significant number of random variables. This strategy will be used in this paper to build up a SPCE of the system response.

The iterative procedure suggested by Blatman and Sudret (2010) for building up a SPCE can be described as follows:

1. Prescribe a target accuracy R^2_{TARGET}, a coefficient q for the hyperbolic truncation scheme, and a maximal order p that the SPCE can reach. In this paper, a target accuracy $R^2_{TARGET} = 0.999$, a coefficient q=0.7, and a maximal order p=5 were used.
2. Consider a set of K realizations (in our case K=200) of the standard normal random vector ξ, called experimental design (ED) and collect the corresponding model evaluations in the vector Γ. Consider also an empty matrix A.
3. Initialization (p=0): add to A the Ψ_0 term corresponding to p=0 and which result from the multiplication of the $H_{\alpha i}$ where all the α_i (i=0, 1, …, M) are equal to zero.
4. Enrichment of the PCE basis (p = p+1): Two sub steps are performed within this step as follows:
   - Forward step: Add to A one by one the different Ψ_β (which have not been considered before) that have a q-norm satisfying $p-1 \leq \|\alpha\|_q \leq p$ and for which a significant increase in the coefficient of determination R^2 is obtained.
   - Backward step: Discard from A the Ψ_β terms [with a q-norm strictly less than p (i.e. $\|\alpha\|_q < p$)] that lead to a negligible decrease in the coefficient of determination R^2.

One should note that the coefficient of determination R^2 is used to check the goodness of fit of the SPCE [Blatman and Sudret (2010)]. The value $R^2=1$ indicates a perfect fit of the true model response Γ, whereas $R^2=0$ indicates a nonlinear relationship between the true model Γ and the SPCE model Γ_{SPCE}.

5. Go to step 4 to perform an enrichment of the (ED) by adding some realizations of the vector ξ (in our case a block of 100 new realizations is added), if the regression problem is ill-posed. Otherwise go to step 6.
6. Stop if either the target accuracy R^2_{TARGET} is achieved or if p reached the order fixed by the user, otherwise go to step 4.

Once the unknown coefficients of the SPCE are determined, the PDF of the ultimate bearing capacity can be estimated. This can be done by simulating a large number of realizations (using Monte Carlo technique) of the standard normal variables on the meta-model.

NUMERICAL RESULTS

The aim of this section is to present the probabilistic results. It should be remembered here that the system response involves the ultimate bearing capacity of a rough rigid strip footing subjected to a symmetrical vertical load. The footing is placed on a weightless spatially varying frictional and cohesive (c, φ) soil with no surcharge loading on the ground surface. The friction angle φ is assumed to follow a beta distribution, and the cohesion c is assumed to be lognormally distributed. The mean values and coefficients of variation of the two random fields are given as follows: $\mu_c = 20kPa, Cov_c = 25\%$; $\mu_\varphi = 30^0, Cov_\varphi = 10\%$. The deterministic model is based on numerical simulations using the finite difference code FLAC^{3D}. The adopted soil domain considered in the analysis is 15m wide by 6m deep. It should be noted that the size of a given element in the mesh depends on the autocorrelation distances of the soil properties. Der Kiureghian and Ke (1988) have suggested that the length of the smallest element in a given direction (horizontal or vertical) should not exceed 0.5 times the autocorrelation distance in that direction. For the boundary conditions, the horizontal movement on the vertical boundaries of the grid is restrained, while the base of the grid is not allowed to move in both the horizontal and the vertical directions. The rough strip footing of 2m width and 0.5m height is assumed to be weightless and it is supposed to follow an elastic linear material. As shown in Fig. 1, the spatial variability of the soil properties can produce (for a given realization) a non-symmetrical mechanism with a footing rotation although the footing is subjected to a symmetrical vertical load.

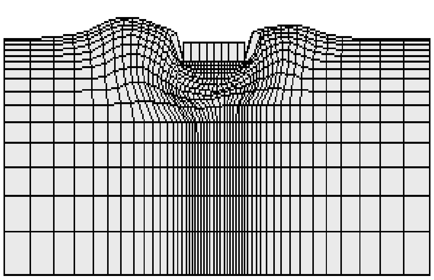

Figure 1. Deformed mesh

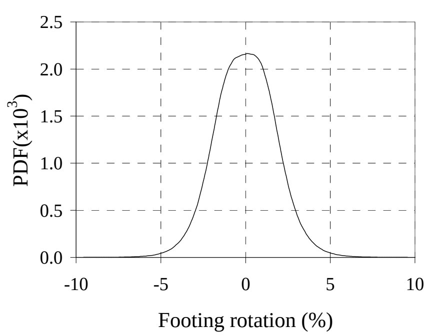

Figure 2. PDF of the footing rotation

Figure 2 presents the PDF of the footing rotation for a reference case considered in this paper for which a_x=10m, a_y=1m, and r(c, φ)=-0.5 where r(c, φ) is the cross-correlation coefficient between the two random fields. This figure shows that the footing rotation of a single realization is not null, but the mean value of all rotations for the whole realizations is null. The standard deviation is found to be equal to $1.6e^{-4}$. In the following sections, one examines the effect of the different probabilistic governing parameters of the two random fields on the PDF of the ultimate bearing capacity of the foundation.

Effect of the autocorrelation distances a_x and a_y

Figure 3 presents the PDF of the ultimate bearing capacity for different values of a_y (a_y=0.5, 0.8, 1, 2, 5, 8m) when a_x=10m and r(c, φ)=-0.5.This figure shows that the PDF is less spread out when the vertical autocorrelation distance a_y decreases. The variability of the ultimate bearing capacity decreases with the increase in the soil heterogeneity since the zone involved by the possible failure surface will have average values of the shear strength parameters close to the mean values of the two fields because of the large number of high and small values of the shear strength parameters. This leads to close values of the ultimate bearing capacity and thus to a smaller variability in the ultimate bearing capacity. Fig. 4 shows that the probabilistic mean value of the ultimate bearing capacity presents a minimum.

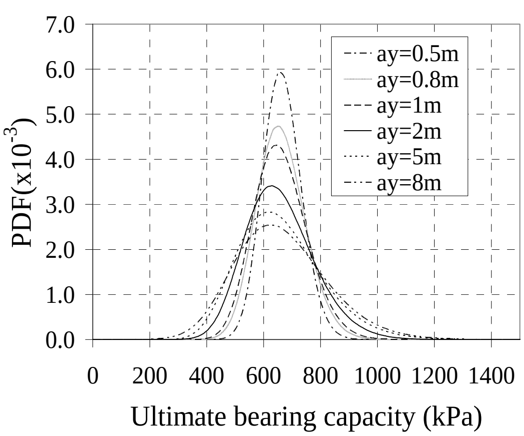

Figure 3. Influence of a_y on the PDF of the bearing capacity in the case where r(c, φ)=-0.5 and a_x=10m

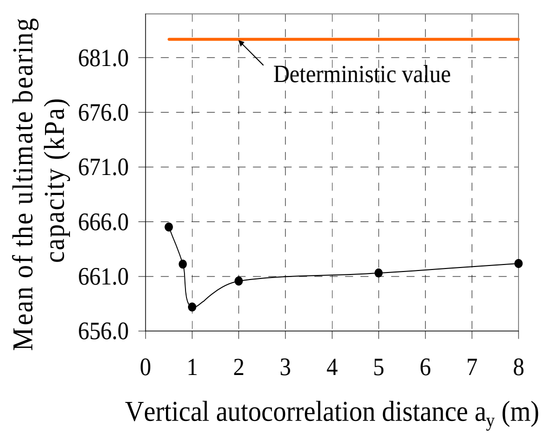

Figure 4. Influence of a_y on the probabilistic mean in the case where r(c, φ)=-0.5 and a_x=10m

This minimum was obtained when a_y=1m, i.e. when the ratio between the horizontal and the vertical autocorrelation distances is equal to 10 for B=2m. When a_y decreases from 8m to 1m, one can notice that the mean ultimate bearing capacity decreases. This can be explained by the fact that increasing the soil heterogeneity introduces weakness zones, thus leading to smaller bearing capacity. The increase in the ultimate bearing capacity for values of a_y smaller than 1m may be explained by the fact that as the autocorrelation distance decreases, the weakest path becomes increasingly tortuous and its length is also longer. As a result, the failure mechanisms will start to look for shorter path cutting through higher values of the shear strength parameters.
Figure 5 presents the PDF of the ultimate bearing capacity for different values of a_x (a_x=2, 4, 10, 20, 30, 50m) when a_y=1m and r(c, φ)=-0.5. The same observations made

before remain valid in the present case. Notice that beyond a value of a_x=20m, the horizontal autocorrelation distance have a small effect on the variability of the ultimate bearing capacity. The minimum probabilistic mean value of the ultimate bearing capacity was obtained at the same ratio a_x/a_y found before [see Fig. 6].

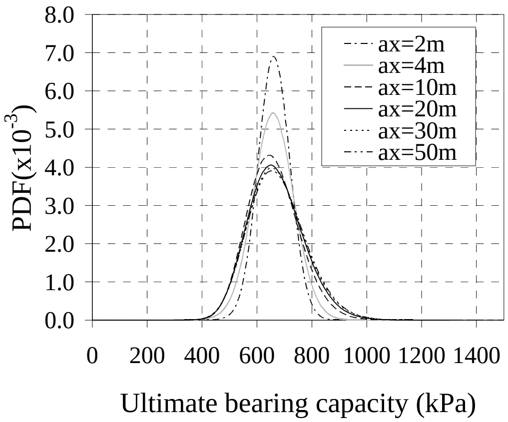

Figure 5. Influence of a_x on the PDF of the bearing capacity in the case where r(c, φ)=-0.5 and a_y=1m

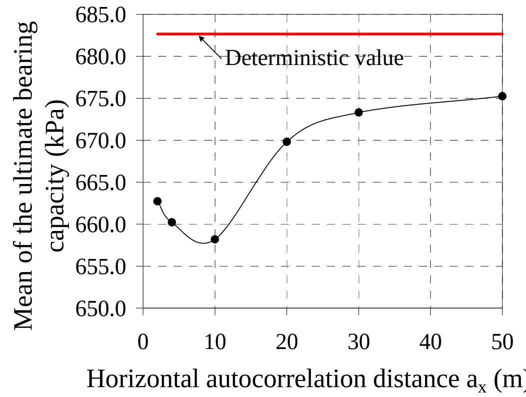

Figure 6. Influence of a_x on the probabilistic mean in the case where r(c, φ)=-0.5 and a_y=1m

Effect of the cross-correlation coefficient and the coefficients of variation of the random fields

Figure 7 presents the PDF of the ultimate bearing capacity for negatively cross-correlated r(c, φ)=-0.5 and uncorrelated r(c, φ)=0 random fields when a_x=10m and a_y=1m. This figure shows that the PDF is less spread out in the case of a negative correlation between the two random fields.

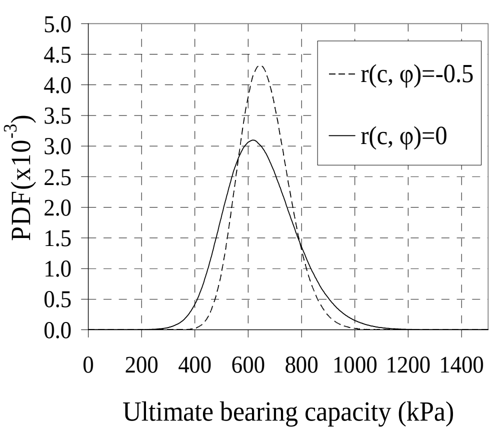

Figure 7. Influence of r(c, φ) on the PDF of the bearing capacity in the case where a_x=10m and a_y=1m

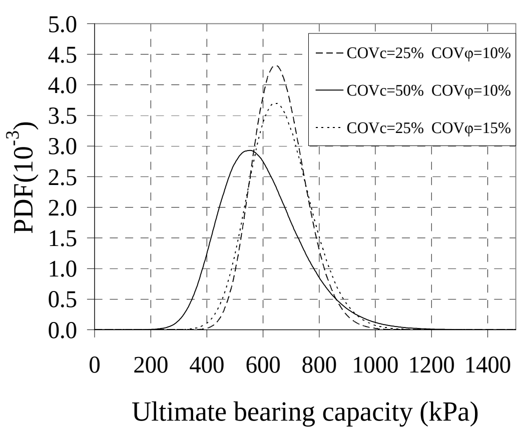

Figure 8. Influence of COV of c and φ on the PDF of the bearing capacity in the case where a_x=10m and a_y=1m

The negative correlation decreases the variability of the response because the increase of one parameter value implies a decrease in the value of the other parameter. Thus, the total shear strength slightly varies. This leads to a reduced variation in the

ultimate bearing capacity. It should be mentioned that the probabilistic mean value of the ultimate bearing capacity slightly increases when a negative correlation between the two random fields exists.
Figure 8 presents the PDF of the ultimate bearing capacity for three configurations of the coefficients of variation. Notice that for the three cases, the cross-correlation coefficient and the autocorrelation distances are those of the reference case where r(c, φ)=-0.5, a_x=10m, and a_y=1m. Fig.8 shows that the PDF becomes more spread out when the coefficients of variation increase. The coefficient of variation of the cohesion parameter was found to have a more significant influence than that of φ on the variability of the system response.

CONCLUSIONS

The effect of the spatial variability of anisotropic cross-correlated non-Gaussian shear strength parameters on the ultimate bearing capacity of a strip footing was studied. The deterministic model was based on numerical simulations using the finite difference code $FLAC^{3D}$. An efficient uncertainty propagation methodology was employed in this paper. This methodology makes use of a non-intrusive approach to build up a sparse polynomial chaos expansion (SPCE) for the system response. This methodology allows one to compute the system response by an analytical expression and then determine the PDF of the system response by simulating a large number of realizations using the Monte Carlo technique. The main conclusions can be summarized as follows: (i) the spatial variability of soil properties produces a non-symmetrical mechanism with a footing rotation although the footing is subjected to a symmetrical vertical load, but the mean value of all rotations for the whole realizations is null; (ii) the probabilistic mean of the ultimate bearing capacity presents a minimum for a given ratio of the horizontal and vertical autocorrelation distances; (iii) the negative correlation between the soil shear strength parameters decreases the response variability; (iv) the variability of the cohesion parameter has a more significant influence than that of the friction angle on the statistical moments of ultimate bearing capacity.

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