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On the Synthesis of a Nominal Trajectory for Control Law of a One-Link Flexible Arm

Abstract

In this article we propose a method for the design of a nominal trajectory for a flexible one-link manipulator. This trajectory is near time-optimal. The torque control law consists of two parts: the commanded feedforward torque and the linear angular position and velocity feedback. Feedforward signal is proportional to nominal angular acceleration of the hub. On the first time interval, the feedforward torque approaches continuously to a value that is a little smaller than the maximal possible torque, then it remains constant, and then it goes to zero. On the second interval, the feedforward torque approaches continuously to a value that is a little greater than the minimal possible torque, then it remains constant, and then it goes to zero. On the third time interval, feedforward signal is zero. We compute the corresponding nominal (desired) angular acceleration, velocity, and position of the hub as functions of time. The last two functions are fed to the linear feedback system. The angular acceleration is such that, on the first time interval, the hub moves with "large" acceleration in one direction, and the link bends on the opposite side. On the second time interval, the hub moves with deceleration, and the link bends in the direction of the motion. On the third time interval we stabilize the arm near its desired position.

In the experiments, the designed control algorithm was successfully implemented. We analyze these experiments from a theoretical point of view.

1. Introduction

The synthesis of a control law for a flexible robot implies contradictory requirements. The response of the system must be fast. If the structure of the model of the controlled object and its parameters are known, it is possible to synthesize the fast control law based on the optimal control theory. However, the model and parameters of the object are often known only approximately. Moreover, the time-optimal problem can be difficult theoretically.

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More importantly, time-optimal control is usually discontinuous (bang-bang control) and, under such a control, large elastic vibrations appear. This is not acceptable for a flexible robot. For these reasons, we must discard time-optimal control. In this work, we design a control law for a flexible one-link arm with some compromise nominal trajectory. First, this control is quasi-time-optimal and, second, it does not produce large elastic oscillations.

The problem of control of a flexible one-link robot has been studied by many investigators (Akulenko and Bolotnik 1982; Berbyuk and Demidyuk 1984; Cannon and Schmitz 1984; Bayo 1987; Bayo et al. 1988; Siciliano and Book 1988; De Luca and Siciliano 1989; Chedmail and Khalil 1989; Lavrovsky and Formal'sky 1989; Pfeiffer 1989; Yuan et al. 1989; Cetinkunt and Wen-Lung 1991; Levis and Vandergrift 1993; Pham et al. 1993; Aoustin et al. 1994, Formal'sky and Lavrovsky 1996). However, to the best of our knowledge, this association of a feedforward and a nominal trajectory has not yet been studied. It seems that the approach described here can be applied to the synthesis of a control for a mechanism with flexible joint (De Luca et al. 1985; Spong 1987; Spong et al. 1987; Isidori 1989; Aoustin 1993), for the crane with hanged load (Chernous'ko et al. 1982) and for other systems for which the number of actuators is smaller than the number of degrees of freedom.

The article is organized as follows. Section 2 contains the statement of the problem. In Section 3, the simple dynamic model of the one-link flexible arm is analyzed. The properties of the system with discontinuous and continuous control are considered. Section 4 presents the distributed parameter model of the arm. The properties of the system with discontinuous and continuous control are examined with this more exact model. In Section 5, we describe the desired nominal regime of the motion of the rigid variable (i.e., the joint angle and the method of its computation). In Section 6, the control law is described. Section 7 contains a short description of the experimental device. Section 8 presents the results of experimental investigations. In Section 9, we analyze theoretically the

The rigid one-link arm is described by equation

$$a_{11}\ddot{\Theta}(t) = \Gamma, \qquad |\Gamma| \leq \Gamma_M. \tag{3}$$

Here, the value

$$a_{11} = J_m + \frac{1}{3}m_b L^2 + ML^2 + J \tag{4}$$

is the moment of inertia of the arm with the motor armature and the load about point 0. It is well known (Pontryagin et al. 1969) that if $\Theta_d > 0$, then the time-optimal control $\Gamma(t)$ for the system (3) can be written in the following form:

$$\Gamma = \begin{cases} \Gamma_M, & \text{if } 0 \leq \Theta \leq \frac{\Theta_d}{2} \\ -\Gamma_M, & \text{if } \frac{\Theta_d}{2} < \Theta < \Theta_d \\ 0, & \text{if } \Theta_d = \Theta \end{cases} \tag{5}$$

The minimal time is (Pontryagin et al. 1969):

$$T = 2\left(\frac{\Theta_d a_{11}}{\Gamma_M}\right)^{1/2}. \tag{6}$$

The time-optimal control (5) and corresponding acceleration contain jumps. In Sections 3 and 4, it is shown that these jumps are not acceptable for the flexible arm because large elastic vibrations of the link appear. It is better to use "fluent" control (Akulenko 1991). This conclusion follows not only from theoretical study, but also from the experiments.

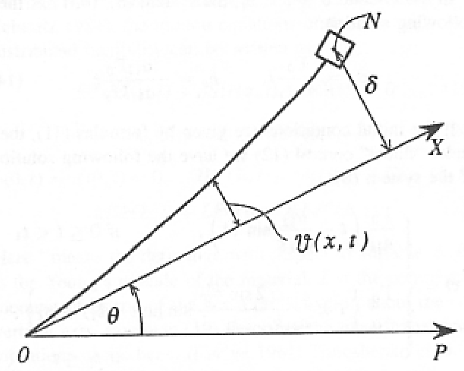

Fig. 1. Scheme of flexible arm.

results of the experiments. Finally, Section 10 contains our conclusion.

2. Statement of the Problem

Consider a flexible homogeneous arm of length L and constant cross section S. Let ρ be the link (beam) material density and $m_b = \rho SL$ its mass. The arm can be rotated in a horizontal plane (the plane of Figure 1) around one of its ends, motionless point O. The other end N is clamped to the center of mass of a load. Let M and J denote, respectively, the mass of this load and its moment of inertia about the mass center N. We show in Figure 1 the (curved) neutral line ON of the beam. The moving coordinate axis OX is tangent to the neutral line ON at point O. Let Θ denote the angle between the axis OX and some motionless direction OP. The motion of the arm is actuated by an electric motor. Let J_m denote the moment of inertia of the motor armature about its axis, Γ is the control torque of electromagnetic forces. The torque Γ is usually limited by some constant Γ_M.

The problem is to find the control Γ that transfers the manipulator from its initial position

$$\Theta(0) = \dot{\Theta}(0) = 0 \tag{1}$$

to an arbitrary final position

$$\Theta(T) = \Theta_d, \ \dot{\Theta}(T) = 0 \tag{2}$$

and keeps it in this position. Here Θ_d is the desired angle of rotation of the arm. The time T is not given, but it is required that this time should be as short as possible. The initial and final elastic deformations are desired to be zero.

3. Simple Mathematical Model

The mathematical models of flexible robots have been considered by many researchers. Using the well-known method of finite number of modes (Book 1984; Cannon and Schmitz 1984; Siciliano and Book 1988; De Luca and Siciliano 1989, 1991, 1993) or a finite element method (Bayo 1987; Chedmail and Khalil 1989; Chedmail et al. 1991) we can design a simple linear mathematical model of the flexible arm,

$$\begin{aligned} a_{11}\ddot{\Theta} + a_{12}\ddot{\delta} &= \Gamma, \\ a_{21}\ddot{\Theta} + a_{22}\ddot{\delta} + k\delta &= 0. \end{aligned} \tag{7}$$

Here, δ is the displacement of point N (Fig. 1). The coefficients a_{12} ($a_{12} = a_{21}$), a_{22} and stiffness coefficient k are positive constants. Numerically, the constants obtained using the method of finite number of modes (Book 1984; Cannon and Schmitz 1984; Siciliano and Book 1988; De Luca and Siciliano 1989, 1991, 1993) or a method developed by Chedmail and Khalil (1989), and Chedmail et al. (1991) are similar, but different.

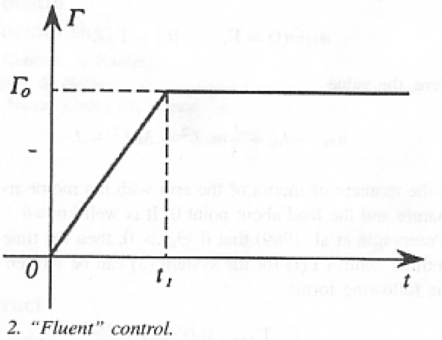

Fig. 2. "Fluent" control.

Equations (7) can be rewritten as:

$$d\ddot{\Theta} - a_{12}k\delta = a_{22}\Gamma$$
$$d\ddot{\delta} + a_{11}k\delta = -a_{12}\Gamma \quad (d = a_{11}a_{22} - a_{12}^2) \tag{8}$$

Let

$$\Gamma = \begin{cases} 0, & \text{if } t < 0 \\ \Gamma_0, & \text{if } t \geq 0 \quad (\Gamma_0 = \text{const}) \end{cases} \tag{9}$$

Under control (9) the system (7) or (8) has a stationary solution:

$$\ddot{\Theta}_s = \frac{\Gamma_0}{a_{11}}, \quad \delta_s = -\frac{a_{12}\Gamma_0}{a_{11}k} \tag{10}$$

In formulas (10), the deflection δ is in opposite phase (De Luca and Siciliano 1989) to the acceleration $\ddot{\Theta}$. If, for example, acceleration $\ddot{\Theta}$ is positive, the deflection δ is negative.

If

$$\delta(0) = \dot{\delta}(0) = 0, \tag{11}$$

the solution of the system (8) under control (9) is:

$$\ddot{\Theta}(t) = \frac{\Gamma_0}{a_{11}}\left(1 + \frac{a_{12}^2}{d}\cos\omega t\right),$$
$$\delta(t) = -\frac{a_{12}\Gamma_0}{a_{11}k}(1 - \cos\omega t) \quad \left(\omega^2 = \frac{a_{11}k}{d}\right) \tag{12}$$

Formulas (12) describe the oscillations of the system near the stationary motion (10).

Consider now, instead of control (9) with a jump of the torque at time $t = 0$, other control which changes continuously (see Fig. 2):

$$\Gamma = \begin{cases} 0, & \text{if } t < 0 \\ \dfrac{\Gamma_0}{t_1}t, & \text{if } 0 \leq t < t_1 \\ \Gamma_0, & \text{if } t \geq t_1 \end{cases} \tag{13}$$

In the interval $0 \leq t < t_1$, the system (8), (13) has the following solution:

$$\ddot{\Theta}_s = \frac{\Gamma_0}{a_{11}t_1}t, \qquad \delta_s = -\frac{a_{12}\Gamma_0}{a_{11}kt_1}t \tag{14}$$

If the initial conditions are given by formulas (11), then under "fluent" control (12) we have the following solution of the system (8):

$$\ddot{\Theta} = \begin{cases} \dfrac{\Gamma_0}{a_{11}}\left(t - \dfrac{a_{12}^2}{a_{11}k}\sin\omega t\right), & \text{if } 0 \leq t < t_1 \\ \dfrac{\Gamma_0}{a_{11}}\left\{1 \pm \dfrac{a_{12}^2\omega}{a_{11}k}\dfrac{2\sin\frac{\omega t_1}{2}}{t_1}\sin\left[\omega(t - t_1) + \varphi\right]\right\}, & \text{if } t_1 \leq t \end{cases} \tag{15}$$

$$\delta = \begin{cases} \dfrac{-a_{12}\Gamma_0}{a_{11}kt_1}\left(t - \dfrac{1}{\omega}\sin\omega t\right), & \text{if } 0 \leq t < t_1 \\ \dfrac{-a_{11}\Gamma_0}{a_{11}k}\left\{1 \pm \dfrac{2\sin\frac{\omega t_1}{2}}{t_1\omega}\sin\left[\omega(t - t_1) + \varphi\right]\right\}, & \text{if } t_1 \leq t \end{cases} \tag{16}$$

$$\left(ctg\varphi = -tg\frac{\omega t_1}{2}\right)$$

Formulas (15) and (16) show that, if the value t_1 is large, then the amplitude of the vibrations of the acceleration $\ddot{\Theta}$ and the deflection δ near the values

$$\ddot{\Theta} = \begin{cases} \dfrac{\Gamma_0}{a_{11}t_1}t, & \text{if } 0 \leq t < t_1 \\ \dfrac{\Gamma_0}{a_{11}}, & \text{if } t_1 \leq t \end{cases} \tag{17}$$

$$\delta = \begin{cases} \dfrac{-a_{12}\Gamma_0}{a_{11}kt_1}t, & \text{if } 0 \leq t < t_1 \\ \dfrac{-a_{12}\Gamma_0}{a_{11}k}, & \text{if } t_1 \leq t \end{cases} \tag{18}$$

is small. Thus, if functions (17) and (18) describe the desired motion, then it is possible, by the choice of the time t_1, to make the vibrations small near this motion, as desired. It is especially relevant for time $t > t_1$.

4. More Exact Mathematical Model

Denote by $v(x, t)$ the deviation of the point with coordinate x of the link neutral line at time t from the moving axis $0X$ (Fig. 1). In the framework of the linear theory of thin straight nonextensible beams (Lur'ye 1961;

Vol'mir 1967; Timoshenko et al. 1974; Cannon and Schmitz 1984), the motion equations of the arm with distributed flexibility can be written as:

$$EIv''''(x,t) + \rho S[\ddot{v}(x,t) + x\ddot{\Theta}(t)] = 0 \quad (19)$$

$$J_m\ddot{\Theta}(t) = \Gamma + EIv''(0,t) \quad (20)$$

$$v(0,t) = v'(0,t) = 0, \quad J[\ddot{v}'(L,t) + \ddot{\Theta}(t)] = -EIv''(L,t),$$
$$M[\ddot{v}(L,t) + L\ddot{\Theta}(t)] = EIv'''(L,t) \quad (21)$$

Here $'$ means the derivative with respect to variable x, E is the Young's module of the material, I is the constant moment of inertia of the beam cross section about the vertical axis. Equation (19) describes the plane transverse vibrations of the beam (Lur'ye 1961; Timoshenko et al. 1974), given the angular acceleration $\ddot{\Theta}$ of the moving axis $0X$. The equation does not take into account the centrifugal force and the energy dissipation during the motion. In the experiments that are described below, the centrifugal force $\rho Sv(x,t)\dot{\Theta}^2$ is essentially smaller than the force $\rho Sx\ddot{\Theta}$, because the end-point deflection $\delta(t) = v(L,t)$ is less than 0.03 m. The drive motion is described by equation (20), the second term on the right-hand side of which is the torque of the forces acting on the armature from the beam. Equations (21) describe the boundary conditions.

Under control torque (9) the system (19)–(21) has the following stationary solution (see analogous solution (10)):

$$\ddot{\Theta}_s = \frac{\Gamma_0}{a_{11}} \quad (22)$$
$$v_s(x,t) = \mathcal{W}(x)$$
$$= \frac{\Gamma_0}{a_{11}EI}\left[-\frac{1}{120}\frac{m_b}{L}x^5 + \frac{L}{6}\left(M + \frac{m_b}{2}\right)x^3 - \frac{1}{2}\left(ML^2 + \frac{m_bL^2}{3} + J\right)x^2\right]$$

It is easy to see that if $\Gamma_0 > 0$, the function $\mathcal{W}(x)$ is negative for all $0 < x \leq L$. This means that if the hub (axis $0X$) rotates counterclockwise with constant acceleration $\ddot{\Theta}$, the link deviates from the axis $0X$ in the other side. If $\ddot{\Theta} < 0$, the function $\mathcal{W}(x)$ is positive for all $0 < x \leq L$. Thus, the deviation of the beam is in opposite phase to the acceleration. For $\Gamma_0 > 0$, the derivative $\mathcal{W}'(x)$ is negative in the interval $0 < x \leq L$.

It can be shown that the general solution of the non-homogeneous system (19)–(21) under control (9) can be expressed in the form:

$$\ddot{\Theta}(t) = \frac{\Gamma_0}{a_{11}} - \sum_{\ell=1}^{\infty} X_\ell'(0)\omega_\ell^2(A_\ell\cos\omega_\ell t + B_\ell\sin\omega_\ell t) \quad (23)$$

$$v(x,t) = \mathcal{W}(x) + \sum_{\ell=1}^{\infty}[X_\ell(x) - xX_\ell'(0)] \times (A_\ell\cos\omega_\ell t + B_\ell\sin\omega_\ell t)$$

Here ω_ℓ (see Section 9) and $X_\ell(x)$ are eigen frequencies and eigen functions of the homogeneous boundary value problem (Cannon and Schmitz 1984; Lavrovsky and Formal'sky 1989)

$$EIu''''(x,t) + \rho S\ddot{u}(x,t) = 0$$
$$J_m\ddot{u}'(0,t) = EIu''(0,t), \quad u(0,t) = 0, \quad (24)$$
$$J\ddot{u}'(L,t) = -EIu''(L,t), \quad M\ddot{u}(L,t) = EIu'''(L,t),$$

where the new variable,

$$u(x,t) = v(x,t) - \mathcal{W}(x) + x[\Theta(t) - \Theta_s(t)]$$

characterizes the difference between the total deviation $v(x,t) + x\Theta(t)$ of the deformed beam from the motionless axis OP and its total stationary deviation $\mathcal{W}(x) + x\Theta_s(t)$ from this axis. System (20) is conservative; therefore, its eigen values λ_ℓ are imaginary, $\lambda_\ell = i\omega_\ell$. The constants A_ℓ, B_ℓ are determined by initial conditions.

If

$$v(x,0) = \dot{v}(x,0) = 0, \quad \Theta(0) = \Theta_s(0), \quad \dot{\Theta}(0) = \dot{\Theta}_s(0), \quad (25)$$

then

$$A_\ell = -\int_0^L \mathcal{W}(x)X_\ell(x)\,dx, \quad B_\ell = 0$$

Formulas (23) describe the vibrations of the system (19)–(21), (9) near the stationary motion (22), as the formulas (12).

Consider now the "fluent" control (13). Under this control, the system (19)–(21) in the interval $0 \leq t < t_1$ has a solution (see analogous solution (14)):

$$\ddot{\Theta}_s(t) = \frac{\Gamma_0}{a_{11}t_1}t, \quad v_s(x,t) = \mathcal{W}(x)\frac{t}{t_1} \quad (26)$$

The general solution of the system (19)–(21), (13) in the interval $0 \leq t < t_1$ can be written in the form:

$$\ddot{\Theta}(t) = \frac{\Gamma_0}{a_{11}t_1}t - \sum_{\ell=1}^{\infty} X_\ell'(0)\omega_\ell^2(A_\ell\cos\omega_\ell t + B_\ell\sin\omega_\ell t) \quad (27)$$

$$v(x,t) = \mathcal{W}(x)\frac{t}{t_1} + \sum_{\ell=1}^{\infty}[X_\ell(x) - xX_\ell'(0)] \times (A_\ell\cos\omega_\ell t + B_\ell\sin\omega_\ell t)$$

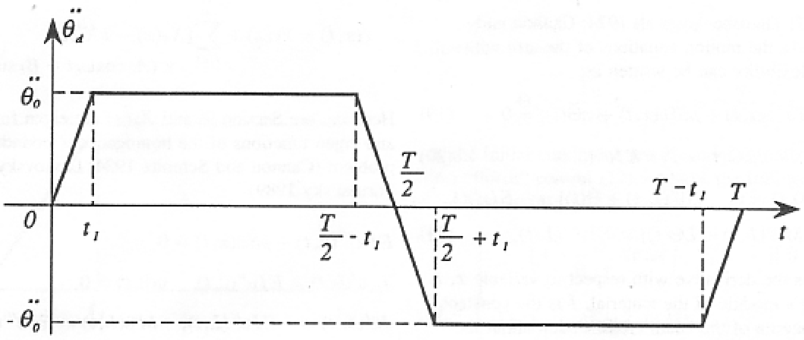

Fig. 3. Desired (nominal) acceleration.

Under conditions (25) we have

$$A_\ell = 0 \quad B_\ell = -\frac{1}{t_1\omega_\ell}\int_0^L \mathcal{W}(x)X_\ell(x)\,dx$$

By choosing time t_1 sufficiently large we could make the vibrations of the system beside the motion (26) sufficiently small. Thus, to the time $t = t_1$ the differences

$$\ddot{\Theta}(t) - \frac{\Gamma_0}{a_{11}}\frac{t}{t_1}, \quad v(x,t) - \mathcal{W}(x)\frac{t}{t_1},$$

and their derivatives with respect to time t would be small, as desired. Therefore, for $t \geq t_1$ the motion of the system (19)–(21), (13) would be near to the motion (22), as desired.

Thus, the consideration of the simplest model in Section 3 and the more exact model in this section shows that the "fluent" control (13) enables the manipulator to reach the desired angular acceleration of the hub and avoid the "large" elastic vibration of the arm. This fact is clear from a physical point of view. Therefore, we use the function (13) for the synthesis of the control law for a flexible one-link arm.

5. Determination of Desired Nominal Regime

We design for $\Theta_d > 0$ the desired nominal acceleration $\ddot{\Theta}_d(t)$ of the flexible one-link arm in the following "trapezoidal" form (see Fig. 3).

$$\ddot{\Theta}_d(t) = \begin{cases} \frac{\ddot{\Theta}_0}{t_1}t, & \text{if } 0 \leq t \leq t_1 \\ \ddot{\Theta}_0, & \text{if } t_1 \leq t \leq \frac{T}{2} - t_1 \\ \frac{\ddot{\Theta}_0}{t_1}\left(\frac{T}{2} - t\right), & \text{if } \frac{T}{2} - t_1 \leq t \leq \frac{T}{2} \end{cases} \tag{28}$$

$$\ddot{\Theta}_d(t) = \begin{cases} -\ddot{\Theta}_d(T-t), & \text{if } \frac{T}{2} \leq t \leq T \\ 0 & \text{if } T \leq t \end{cases}$$

Here

$$\ddot{\Theta}_0 = \frac{\Gamma_0}{a_{11}}$$

is a constant, which is smaller than the maximal possible acceleration

$$\frac{\Gamma_M}{a_{11}}\,(\Gamma_0 < \Gamma_M).$$

By integrating the expression (28) we can obtain the analytical expression for the desired velocity $\dot{\Theta}_d(t)$. It is obvious that $\dot{\Theta}_d(T) = 0$. By integrating the function $\dot{\Theta}_d(t)$, it is easy to write the analytical expression of the desired position $\Theta_d(t)$. Let the final position Θ_d be given, as well as the acceleration $\ddot{\Theta}_0$ and the time t_1. Then it is possible to compute numerically the time T for which the relation $\Theta_d(T) = \Theta_d$ is true. After this, we can design nominal acceleration $\ddot{\Theta}_d(t)$, velocity $\dot{\Theta}_d(t)$ and position $\Theta_d(t)$. We calculate the torque feedforward as:

$$\Gamma_d = a_{11}\ddot{\Theta}_d(t). \tag{29}$$

If the angle Θ_d is small, the acceleration $\ddot{\Theta}_d(t)$ can have the other form (Fig. 4):

$$\ddot{\Theta}_d(t) = \begin{cases} \frac{\ddot{\Theta}_0}{t_1}t, & \text{if } 0 \leq t \leq t_1 \\ \frac{\ddot{\Theta}_0}{t_1}(2t_1 - t), & \text{if } t_1 \leq t \leq 2t_1 \end{cases} \tag{30}$$

$$\ddot{\Theta}_d(t) = \begin{cases} -\ddot{\Theta}_d(4t_1 - t), & \text{if } 2t_1 \leq t \leq 4t_1 \\ 0, & \text{if } 4t_1 \leq t \end{cases}$$

Here $4t_1 = T$.

Observe that torque control (29) with acceleration (28) or (30) is time-optimal control for the boundary value

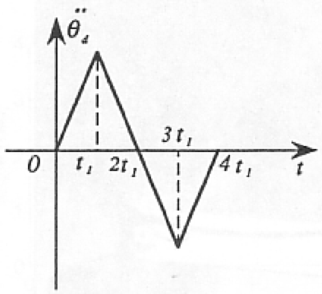

Fig. 4. Desired (nominal) acceleration for "little" Θ_d.

problem (1), (2) with the following motion equation and torque constraints:

$$a_{11}\ddot{\Theta} = \Gamma, \quad |\Gamma| \leq \Gamma_0 \ (\Gamma_0 = a_{11}\ddot{\Theta}_0),$$

$$|\dot{\Gamma}| \leq \dot{\Gamma}_0, \ \left(\dot{\Gamma}_0 = \frac{\Gamma_0}{t_1}\right)$$

6. Control System

If the final angular position Θ_d is given, the computer calculates at first the time T. During the motion, it calculates the commanded values $\Gamma_d(t), \Theta_d(t), \dot{\Theta}_d(t)$ and control torque $\Gamma(t)$, which consists of linear position and velocity feedback, combined with feedforward $\Gamma_d(t)$ (Cannon and Schmitz 1984; De Luca et al. 1985; Bayo 1987; Bayo et al. 1988; Spong 1987; Lavrovsky and Formal'sky 1989; Aoustin and Chevallereau 1993; Aoustin et al. 1994).

$$\Gamma = \beta_1[\Theta_d(t) - \Theta] + \beta_2[\dot{\Theta}_d(t) - \dot{\Theta}] + \Gamma_d(t) \tag{31}$$

The feedback gain β_2 is constant, but the gain β_1 is first constant, then continuously increases, and then is constant again:

$$\beta_1 = \begin{cases} \beta_{10}, & \text{if } 0 \leq t \leq T \\ \beta_{10} + \beta(t - T), & \text{if } T \leq t \leq T + \Delta T \\ \beta_{10} + \beta\Delta T, & \text{if } T + \Delta T \leq t \end{cases} \tag{32}$$

The increase of the coefficient β_1 enables us to decrease the static error. Due to fluent increase of this coefficient, we avoid a jump in the control torque.

Note that in the work by Lavrovsky and Formal'sky (1989, 1996), it is proved for the case when $J = 0$ that the state,

$$v(x, t) \equiv 0, \quad \Theta(t) \equiv \Theta_d = \text{const},$$

is an asymptotically stable state of the system (19)–(21) with closed-loop control

$$\Gamma = \beta_1(\Theta_d - \Theta) - \beta_2\dot{\Theta},$$

if the coefficients β_1, β_2 are positive constants.

7. Experimental Device

In Figure 5, the experimental planar one-link arm is shown. The parameters of this arm are the following:

$$L = 1.005\,\text{m}, \quad m_b = 2.04\,\text{kg}, \quad M = 6.79\,\text{kg}, \\ J = 0.047\,\text{kg}\cdot\text{m}^2, \\ J_m = 0.0018\,\text{kg}\cdot\text{m}^2, \quad EI = 47.25\,\text{N}\cdot\text{m}^2, \\ \Gamma_M = 3.5\,\text{N}\cdot\text{m} \tag{33}$$

The lowest frequency of the vibrations of our robot with cantilever clamped hub is approximately 0.72 Hz. The axis of the rotational joint is vertical. The motion of the arm is controlled by a torque motor. The extremity of the arm has air bearing to avoid gravity effects and, as much as possible, friction between the arm and the horizontal table. The arm is equipped with an optical sensor that measures the angle Θ in the joint and with two deformation sensors (tensometrical sensors) that enable evaluation of the deflection δ of the arm end-point N. We obtain the angular velocity $\dot{\Theta}$ by using numerical calculation and filter. Using the mechanical characteristics (33) of the manipulator, we can calculate its whole moment of inertia (4) $a_{11} = 7.59\,\text{kg}\cdot\text{m}^2$. From the experiments we have obtained $a_{11} = 7.0\,\text{kg}\cdot\text{m}^2$. This value we have used for calculating the feedforward torque (29) in the experiments with control law (31).

8. Experiments

By the experiments we have found that if $\ddot{\Theta}_0/t_1 \leq 1.0\,\text{s}^{-3}$, then the amplitude of elastic vibrations is not large. The inclination $\ddot{\Theta}_0/t_1$ has been chosen $1.0\,\text{s}^{-3}$. We have used in the experiments the value $\ddot{\Theta}_0 = 0.48\,\text{s}^{-2}$. For this value, $a_{11}\ddot{\Theta}_0 = 3.36\,\text{N}\cdot\text{m}$. This product is smaller than the maximum torque $\Gamma_M = 3.5\,\text{N}\cdot\text{m}$. Other parameters of control law (31) are the following:

$$\beta_{10} = 10, \quad \beta = 5\,\text{s}^{-1}, \quad \Delta T = 1\,\text{s}, \quad \beta_2 = 2\,\text{s}^{-1}$$

In Figure 6A–H for $\Theta_d = 3$ the angle $\Theta_d(t)$, the desired angle $\Theta_d(t)$, the tracking error $\Theta(t) - \Theta_d(t)$, the velocity $\dot{\Theta}(t)$, the desired velocity $\dot{\Theta}_d(t)$, the desired acceleration $\ddot{\Theta}_d(t)$, the elastic deflection $\delta(t)$ of the arm end, and the torque $\Gamma(t)$, respectively, are displayed. In

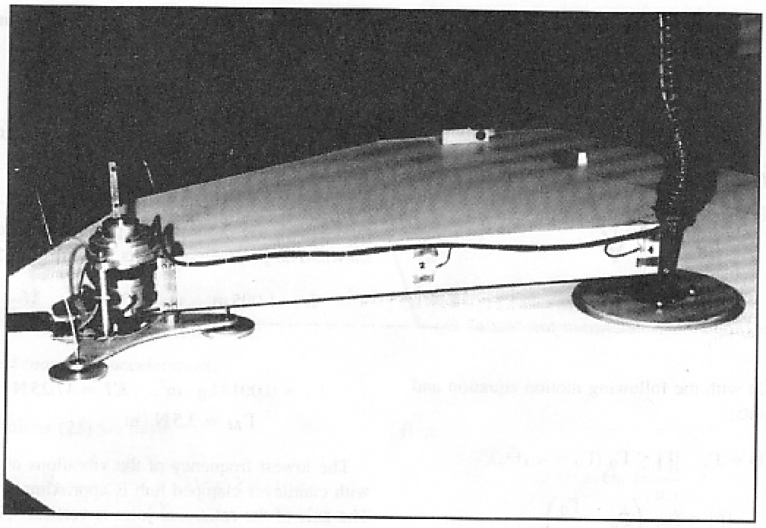

Fig. 5. The experimental arm.

the desired trajectory, the time T of the transitional process is 5.5 s. The optimal time, calculated from formula (6), is 4.9 s. This time is smaller than time 5.5 s by 11%. The difference $|\Theta(t) - \Theta_d(t)|$ is smaller than 0.04 and the angle $\Theta(t)$ tracks the desired angle $\Theta_d(t)$ correctly. Small static error exists at the end of the tracking process. The deflection $|\delta(t)|$ is smaller than 0.03 m. In control law (31) the term with displacement δ is absent, because the elastic vibrations with high frequency are not large, and it is not required to damp them especially. We can decrease the amplitude of high-frequency vibrations by decreasing the value $\ddot{\Theta}_0/t_1$, but the time of transitional process would be greater in this case.

In Figure 7*A–H*, the behavior of the same variables is displayed for $\Theta_d = 0.5$. The static error at the end of this tracking process is close to 0.01. It is possible to decrease the static error by adding to torque (31) the torque

$$\Gamma_f = \begin{cases} \Gamma' \mathrm{sgn}\Theta_d, & \text{if } 0 \le t \le T \\ 0, & \text{if } T < t \end{cases}$$

that can compensate for the torque of the friction forces. By the experiments, we estimated this torque and used the constant value $\Gamma' = 0.1\,\mathrm{N \cdot m}$.

For the other given values Θ_d the transitional processes are similar to those shown in Figures 6 and 7. Some experiments were recorded on film.

For changing the final position Θ_d of the arm, we need only change the parameter Θ_d in the control program. The program itself computes the time T, the desired acceleration $\ddot{\Theta}_d(t)$, velocity $\dot{\Theta}_d(t)$, position $\Theta_d(t)$, and feedforward torque $\Gamma_d(t)$.

9. Theoretical Analysis of the Experiments

Figures 6*G* and 7*G* show that the deflection δ changes mainly in the opposite phase (De Luca and Siciliano 1989) to the feedforward torque (desired acceleration) and has one oscillation. The frequency of this oscillation is defined by the frequency of the desired acceleration. It seems that we can describe this main motion by using the function $\mathcal{W}(x)$ (see formulas (22), (26)) that present the stationary solution of the system (19)–(21). It is correct evidently to use the stationary function $\mathcal{W}(x)$ for the design of the model of the kind (7), assuming that the deflection $\mathcal{W}(L) = \delta$ is the new variable (elastic). We have derived the models of the kind (7), using, first, the function $\mathcal{W}(x)$ and, second, the eigen function (De Luca and Siciliano 1989), corresponding to the first mode of the boundary value problem (19), (21) for the flexible beam with cantilever clamping of the hub ($\Theta(t) \equiv 0$). It occurs that the numerical values of the coefficients a_{12}, a_{22}, k for parameters (33) are "practically" the same in both cases. We can discern the normalized ($\mathcal{W}(L) = 1$) function $\mathcal{W}(x)$ and the normalized first eigen function, inspecting them in "large" scale only.

We see in Figures 6*G* and 7*G* the vibrations of the arm end point with a high frequency that is equal to approximately 6 Hz. (We can find this frequency by calculating

V

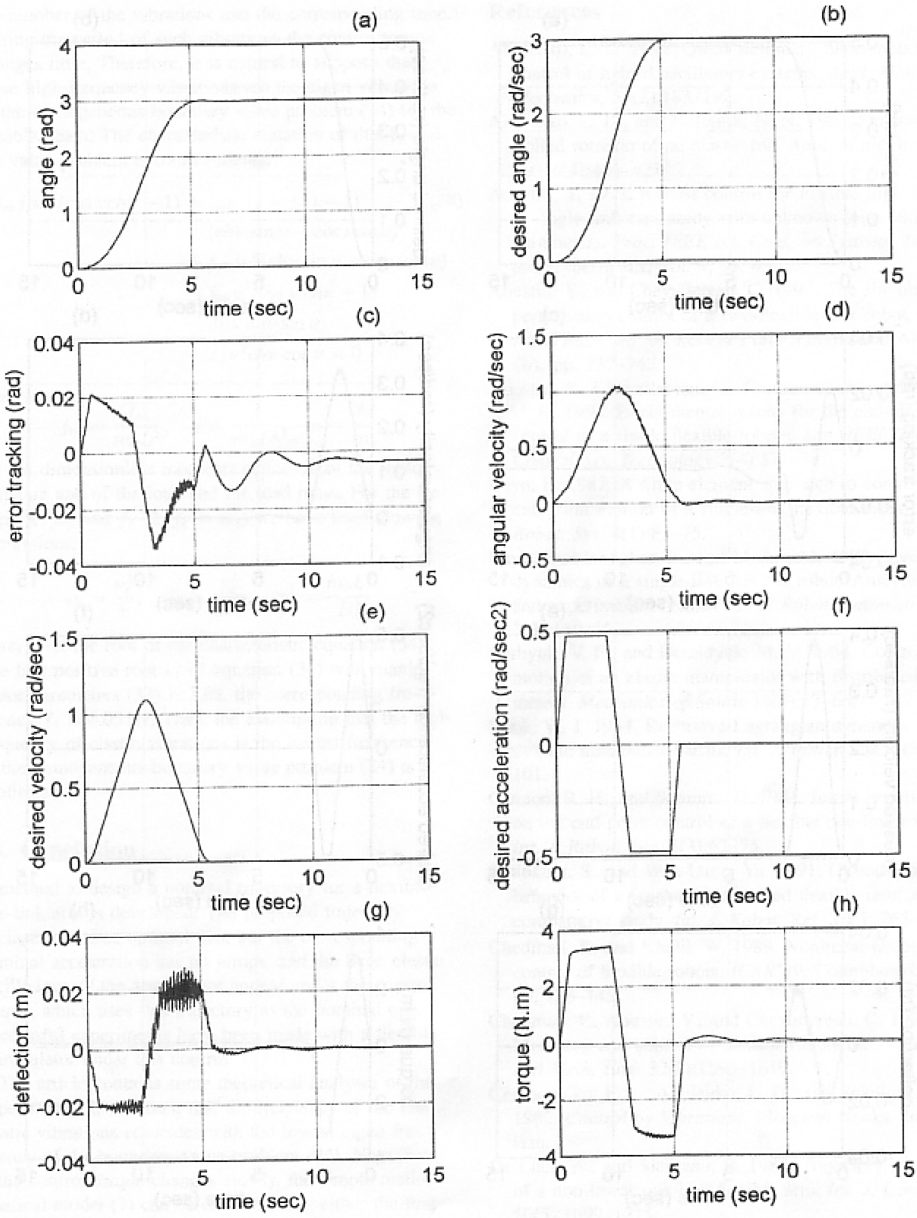

Fig. 6. Experimental results for $\Theta_d = 3$. *A, Angle* $\Theta(t)$. *B, Desired angle* $\Theta_d(t)$. *C, Tracking error* $\Theta(t) - \Theta_d(t)$. *D, Angular velocity* $\dot{\Theta}(t)$. *E, Desired angular velocity* $\dot{\Theta}_d(t)$. *F, Desired acceleration* $\ddot{\Theta}_d(t)$. *G, Deflection* $\delta(t)$. *H, Torque* $\Gamma(t)$.

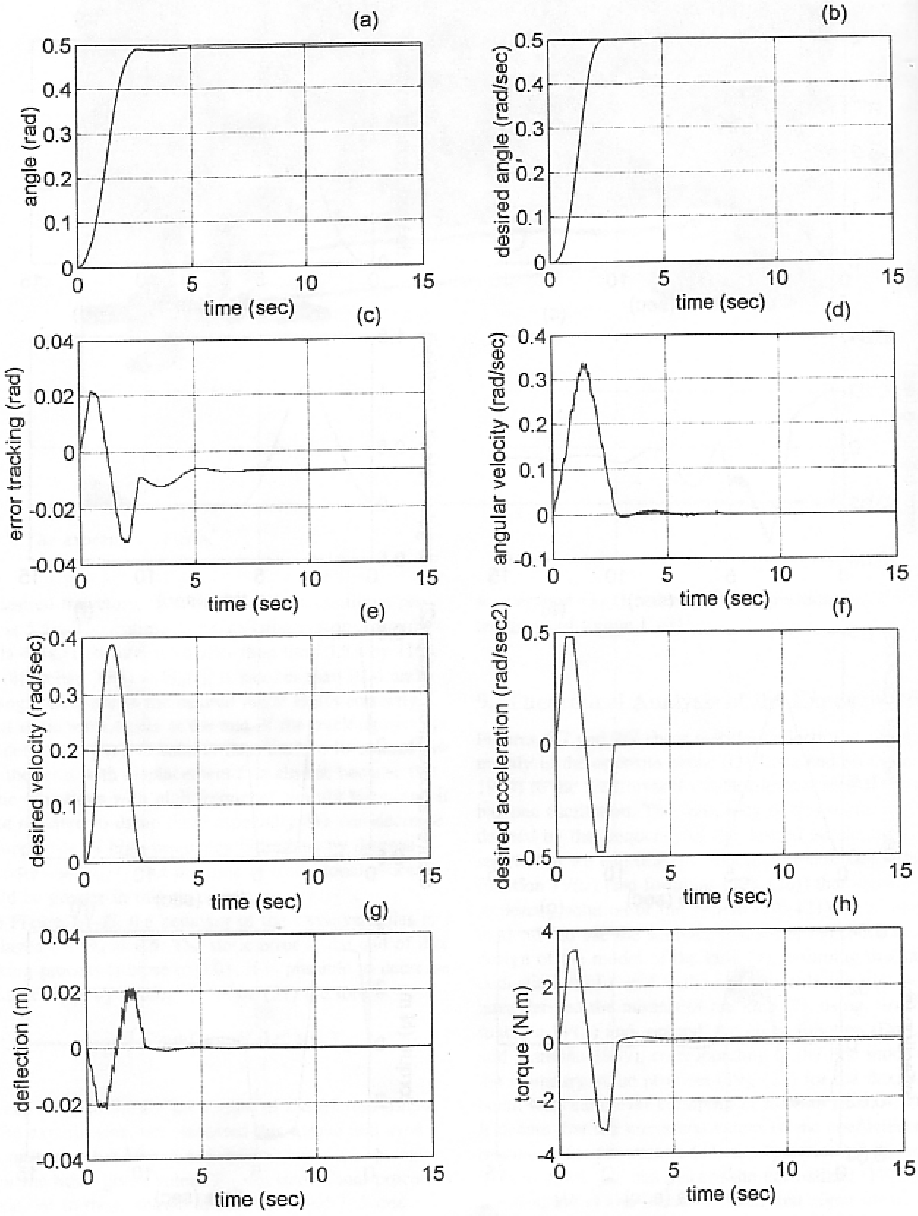

Fig. 7. Experimental results for $\Theta_d = 0.5$. A, Angle $\Theta(t)$. B, Desired angle $\Theta_d(t)$. C, Tracking error $\Theta(t) - \Theta_d(t)$. D, Angular velocity $\dot{\Theta}(t)$. E, Desired angular velocity $\dot{\Theta}_d(t)$. F, Desired acceleration $\ddot{\Theta}_d(t)$. G, Deflection $\delta(t)$. H, Torque $\Gamma(t)$.

V

the number of the vibrations and the corresponding time.) During the period of such vibrations the control torque changes little. Therefore, it is natural to suppose that these high-frequency vibrations are the eigen vibrations of the homogeneous boundary value problem (24) for the flexible beam. The characteristic equation of this boundary value problem is transcendental,

$$\begin{aligned} j_m j \mu \nu^7 (\cos \nu ch\nu - 1) + [\mu \nu^4 (j + j_m) - 1] \\ \times (ch\nu \sin \nu - \cos \nu\, sh\nu) \\ + j_m j \nu^6 (ch\nu \sin \nu + \cos \nu sh\nu) \\ - j_m \nu^3 (\cos \nu\, ch\nu + 1) \\ - 2\mu\nu\, sh\nu \sin \nu \\ - 2j\nu^3 ch\nu \cos \nu = 0 \end{aligned} \tag{34}$$

Here

$$j_m = \frac{J_m}{m_b L^2}, \quad j = \frac{J}{m_b L^2}, \quad \mu = \frac{M}{m_b}$$

are the dimensionless moments of inertia of the motor armature and of the load and the load mass. For the frequencies ω_ℓ and f_ℓ ($2\pi f_\ell = \omega_\ell$) we have the following expressions:

$$\omega_\ell = \frac{\nu_\ell^2}{\tau}, \quad f_\ell = \frac{\nu_\ell^2}{2\pi\tau}, \quad \tau^2 = \frac{m_b L^3}{EI}$$

Here, ν_ℓ is the root of the characteristic equation (34). The first positive root ν_1 of equation (34) with manipulator parameters (33) is 2.82, the corresponding frequency f_1 is 6.05 Hz. Thus, the assumption that the high frequency of elastic vibrations is the lowest frequency of the homogeneous boundary value problem (24) is confirmed.

10. Conclusion

A method to design a nominal trajectory for a flexible one-link arm is developed. The proposed trajectory is close to a time-optimal one, but the corresponding nominal acceleration has no jumps, and the large elastic oscillations of the arm do not appear under the control torque, which uses this trajectory as the nominal one. Successful experiments have been made with a flexible manipulator under this control.

The article contains some theoretical analyses of the experiments. It is shown that the frequency of the fast elastic vibrations coincides with the lowest eigen frequency of the boundary value problem (24). Note that if the control torque changes slowly, the simple mathematical model (7) can be derived using either the first mode of the flexible beam with cantilever clamping of the hub (De Luca and Siciliano 1989), or the stationary solution (22) of the boundary value problem (19)–(21) with $\Gamma = \text{const}$.

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