Finite Time Stabilization of a Perturbed Double Integrator – Part I: Continuous Sliding Mode-based Output Feedback Synthesis

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***Abstract*—The twisting and supertwisting algorithms, generating important classes of second order sliding modes (SOSM's), are well-recognized for their finite time stability and robustness properties. In the present paper, a continuous modification of the twisting algorithm and an inhomogeneous perturbation of the supertwisting algorithm are introduced to extend the class of SOSM's that present the afore-mentioned attractive features. Thus modified, the twisting and supertwisting algorithms are utilized in the state feedback synthesis and, respectively, velocity observer design, made for the finite time stabilization of a double integrator if only output measurements are available. Performance and robustness issues of the resulting output feedback synthesis are illustrated by means of numerical simulations.**

I. INTRODUCTION

Motivated by modern applications, particularly, to electromechanical systems (see, e.g., [1], [2] and references therein), the systematic nonsmooth feedback design methodology is continuing to be developed. Since this methodology is capable of imposing the finite time stability as well as desired robustness properties on the closed-loop system, it has been widely used in practice. An example of such an application is orbital synthesis of hybrid systems (e.g., cyclic bipedal locomotion [3]), where the controlled plant is operating under uncertainties and it is to be driven to an operational periodic mode in finite time during its continuous phase between successive discrete dynamics.

In the present paper, a modification of the twisting controller from [4] and that of the supertwisting observer from [5] are coupled together to present a unified framework for the finite time output feedback stabilization of a double integrator. The afore-mentioned twisting and supertwisting algorithms generate an important class of the so-called second order sliding modes (SOSM's), which are well-recognized for their finite time stability and robustness properties [6], [7].

The former modification represents a parameterized family of homogeneous continuous controllers and it is made for avoiding the undesired chattering phenomenon that appears in the closed-loop if driven by a switching input of high frequency [8], [9]. In turn, observer switching is not as serious as controller switching because it does not drive an actuator and hence it does not come at the expense of controller switching. Therefore, the latter modification represents a parameterized family of inhomogeneous switched observers, and the motivation behind this modification is in extending the class of finite time stable SOSM's towards their inhomogeneous perturbations.

Being coupled together, the modification of the twisting controller and that of the supertwisting observer yield the output feedback stabilization of the double integrator in finite time in accordance with the separation principle which proves to be in force in the present case. The resulting output feedback synthesis constitutes the main theoretical contribution of the paper. The stability analysis of the closed loop system is made within the nonsmooth Lyapunov methods recently developed in [2, Chaptet 3] for discontinuous systems (see [10] for an alternative Lyapunov stability analysis of nonsmooth systems). Finite time convergence and robustness properties of the proposed synthesis are supported by simulations.

II. FINITE TIME STABILIZING STATE FEEDBACK SYNTHESIS

To begin with, we present a class of static feedback controllers that globally stabilize the double integrator

$$\dot{x} = y,\ \dot{y} = u. \tag{1}$$

A feedback law $u(x, y)$ is further referred to as finite time stabilizing if it renders the origin of the closed-loop system (1) a finite time stable equilibrium as defined in [1], [4].

The following state feedback

$$u = -\mu|y|^{\alpha} sign\ y - \nu|x|^{\frac{\alpha}{2-\alpha}} sign\ x \tag{2}$$

with parameters $\nu > \mu > 0$ and $\alpha \in [0, 1)$ is proposed to globally stabilize the double integrator (1). If specified with $\alpha = 0$, the above controller is inspired from the twisting algorithm [6], [7] and it coincides with the discontinuous finite time stabilizing controller of [4], whereas for $\alpha \in (0, 1)$, it simplifies the continuous controller of [1], while also presenting the finite time stability. The proposed simplification is essential both for the stability analysis of the closed-loop system and for improving its performance.

Let us along with the nominal model of the double integrator, consider its perturbed version

$$\dot{x} = y,\ \dot{y} = u + w \tag{3}$$

and let us investigate robustness properties of the closed-loop system (2), (3) against an external disturbance $w(x, y, t)$, being a locally integrable function on all potential trajectories

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$x(t), y(t)$. If $\alpha = 0$, then according to [4, Theorem 4.2], the disturbed system (2), (3) renders the finite time stability, regardless of whichever disturbance w with a uniform upper bound

$$ess \sup_{t\geq 0} |w(t)| \leq M \tag{4}$$

affects the system provided that

$$0 < M < \mu < \nu - M. \tag{5}$$

This robustness property is achieved due to the high frequency controller switching in the sliding mode of the second order that occurs in the origin. In turn, only external disturbances, vanishing in the origin and satisfying the growth condition

$$|w(x, y, t)| \leq \mu_0 |y|^\alpha \tag{6}$$

for some $\alpha \in (0, 1)$, some upper bound $\mu_0 < \mu$, and for all $x, y \in R,\ t \geq 0$, can be rejected by the continuous controller (2) with the same α as that in (6). The following result is in force.

Theorem 1: Given $\alpha \in (0, 1)$ and $\mu, \nu > 0$, the continuous closed-loop system (2), (3) is globally asymptotically stable for any disturbance w, satisfying the growth condition (6) for an arbitrary $\mu_0 < \mu$.

Proof: Under conditions of the theorem, the time derivative of the Lyapunov function

$$V(x, y) = \nu \frac{2-\alpha}{2} |x|^{\frac{2}{2-\alpha}} + \frac{1}{2} y^2, \tag{7}$$

computed along the trajectories of (2), (3), is estimated as follows:

$$\dot{V} \ = \ -\mu |y|^{\alpha+1} + yw \leq -(\mu - \mu_0)|y|^{\alpha+1}. \tag{8}$$

Since $\mu > \mu_0$ by a condition of the theorem, the global asymptotic stability of (2), (3) is then established by applying the invariance principle. ■

Remark 1: Analyzing the proof of Theorem 1, one can note that under non-vanishing disturbances (4), the closed-loop system (2), (3) is practically stabilized in finite time to the vicinity of the radius $(M/\mu_0)^{1/\alpha}$ rather than to the point of interest. Indeed, to conclude this, it suffices to observe that in this case, the time derivative (8) of the Lyapunov function (7) remains negative semidefinite beyond the afore-mentioned vicinity. Thus, the higher μ_0, specified in the growth condition (6), the better the stabilization precision which is attained for the perturbed system (3), affected by a non-vanishing disturbance (4). Meanwhile, the higher the μ_0 employed, the higher the control magnitude μ needed in the control law (2). This is apparently the price one should pay for smoothing the switched controller (2) that corresponds to $\alpha = 0$.

III. Finite Time Velocity Observer Design

In the present section, we focus our study on the stability analysis of the velocity observer

$$\begin{aligned} \dot{\hat{x}} &= \hat{y} + k_1 |x - \hat{x}|^\varepsilon \text{sign } (x - \hat{x}) + k_2(x - \hat{x}) \\ \dot{\hat{y}} &= u + k_3 \text{sign } (x - \hat{x}) + k_4(x - \hat{x}) \end{aligned} \tag{9}$$

of the double integrator (1), which is parameterized with $k_1, k_3 > 0$, $\varepsilon \geq 0$ and which is obtained through a certain perturbation of the supertwisting observer

$$\begin{aligned} \dot{\hat{x}} &= \hat{y} + k_1 |x - \hat{x}|^{\frac{1}{2}} \text{sign } (x - \hat{x}) + k_2(x - \hat{x}) \\ \dot{\hat{y}} &= u + k_3 \text{sign } (x - \hat{x}) + k_4(x - \hat{x}) \end{aligned} \tag{10}$$

that was first proposed in [5] with $k_2, k_4 = 0$ and then augmented in [11] with nontrivial linear gains $k_2, k_4 > 0$. As the above equations (9), (10) possess discontinuous right-hand sides, the dynamics of such equations are throughout defined in the sense of Filippov [12].

Clearly, the observation error $e = (e_1, e_2)^T,\ e_1 = x - \hat{x},\ e_2 = y - \hat{y}$ between the state of the double integrator (1) and that of the velocity observer (9) is governed by the following second-order system

$$\begin{aligned} \dot{e}_1 &= e_2 - k_1 |e_1|^\varepsilon sign\ e_1 - k_2 e_1, \\ \dot{e}_2 &= -k_3 sign\ e_1 - k_4 e_1. \end{aligned} \tag{11}$$

We are in a position to establish the global asymptotic stability of the error system within the framework of methods of nonsmooth Lyapunov functions with negative semidefinite time derivative along the dynamics of the system.

Theorem 2: Let the parameters of system (11) be such that $k_1, k_3 > 0$, $k_2, k_4 \geq 0$, and $\varepsilon \geq 0$. Then system (11) is globally asymptotically stable.

Proof: First of all, let us note that the discontinuous system (11), being viewed in the sense of Filippov, possesses an equilibrium point in the origin. Next let us observe that for (11), the well-known condition $e_1 \dot{e}_1 < 0$ [13] for sliding modes to exist on the switching line $e_1 = 0$ holds for all infinitesimal $e_1 \neq 0$ if and only if $\varepsilon = 0$ and $|e_2| < k_1$. Indeed, $e_1 \dot{e}_1 = e_1(e_2 - k_1 \text{sign } e_1 - k_2 e_1) = -|e_1|(k_2|e_1| + k_1 - e_2 \text{sign } e_1) < 0$ provided that $\varepsilon = 0$ and $|e_2| < k_1$; otherwise, the value of $e_1 \dot{e}_1 = e_1(e_2 - k_1 |e_1|^\varepsilon \text{sign } e_1 - k_2 e_1)$ becomes positive for all infinitesimal $e_1 \neq 0$ whenever $e_1 e_2 > 0$. Thus, in system (11), the sliding modes $e_1(t) = 0$ can only appear and they do appear within the interval $|e_2| < k_1$ just in the case of $\varepsilon = 0$.

Now, let us consider the Lyapunov candidate function

$$V_0(e_1, e_2) = k_3 |e_1| + \frac{1}{2} k_4 e_1^2 + \frac{1}{2} e_2^2. \tag{12}$$

By computing the time derivative of this function along the trajectories of system (11), we arrive at

$$\begin{aligned}\dot{V}_0 &= (k_3\text{sign } e_1 + k_4 e_1)(e_2 - k_1|e_1|^{\varepsilon}\text{sign } e_1 - k_2 e_1)\\ &\quad -e_2(k_3\text{sign } e_1 + k_4 e_1)\\ &= -k_1k_3|e_1|^{\varepsilon} - k_1k_4|e_1|^{\varepsilon+1}\\ &\quad -k_2k_3|e_1| - k_2k_4e_1^2 < 0\end{aligned} \tag{13}$$

everywhere but on the vertical axis $e_1 = 0$ where the function V is not differentiable.

Inequality (13) subject to $\varepsilon = 0$ ensures that for $\varepsilon = 0$, the trajectories of (11) hit the sliding mode interval $I_{k_1} = \{(e_1, e_2)] \in R^2 : e_1 = 0, |e_2| < k_1\}$ in finite time because otherwise they steer to the origin in finite time. Since the sliding modes on the interval I_{k_1} are governed by the asymptotically stable equation

$$\dot{e}_2 = -\frac{k_3}{k_1}e_2, \tag{14}$$

system (11), corresponding to $\varepsilon = 0$, is globally asymptotically stable.

For the convenience of the reader, recall that the sliding mode equation (14) is derived according to the equivalent control method [13] by substituting the equivalent value $\text{sign}_{eq}e_1$ of the commuting function sign e_1 (that ensures the identity $\dot{e}_1 = 0$ along the sliding modes) into the second equation of the disturbance-free system (11) for sign e_1. As a matter of fact, the equivalent value represents a solution of the algebraic equation $e_2 - k_1\text{sign } e_1 = 0$ with respect to sign e_1 and hence, $\text{sign}_{eq}e_1 = k_1^{-1}e_2$.

Finally, confining our demonstration to the case $\varepsilon > 0$ yields no sliding modes on the discontinuity manifold $e_1 = 0$. Thus, by virtue of $\varepsilon > 0$, inequality (13) holds almost everywhere, and the global asymptotic stability in this case is concluded by applying the extension of the invariance principle, made in [2, Theorem 3.2], to the discontinuous system (11). Theorem 2 is completely proved. ■

It should be pointed out that by straightforward inspection [2, Definition 4.6], system (11), specified with $\varepsilon = \frac{1}{2}$ and $k_2 = k_4 = 0$, is homogeneous of degree $q = -1$ with respect to dilation $r = (2, 1)$ so that by the homogeneity principle [2, Theorem 4.2], the global finite time stability of this system is additionally guaranteed by its global asymptotic stability. Our next aim is to demonstrate that the finite time stability of system (11) persists for positive values of the parameters k_2 and k_4 in spite of certain perturbations of the parameter ε around its homogeneity value $\varepsilon = \frac{1}{2}$. For this purpose, we introduce the modified Lyapunov function

$$V_1(e_1, e_2) = 2k_3|e_1| + k_4e_1^2 + \frac{1}{2}e_2^2 + \frac{1}{2}s^2(e_1, e_2) \tag{15}$$

which is obtained by doubling the first two terms of (12) and by augmenting (12) with a half square of the right hand-side

$$s(e_1, e_2) = e_2 - k_1|e_1|^{\varepsilon} sign\ e_1 - k_2e_1, \tag{16}$$

of the first equation of (11). The proposed modification is inspired from [11] where the same Lyapunov function, being specified with $\varepsilon = \frac{1}{2}$, has been used to prove the system finite time stability in this particular case.

As the Lyapunov function (15) is shown to admit the estimate

$$\dot{V}_1(t) \le -\kappa V_1^{\varepsilon}(t) \tag{17}$$

along the trajectories of system (11), specified with $\varepsilon \in [\frac{1}{2}, 1)$, for some $\kappa > 0$ and for almost all $t \ge 0$, it plays a crucial role in establishing the finite time stability of such a system and in determining an upper bound of the settling-time function $T(e_1^0, e_2^0)$ in terms of the initial value $v_0 = V_1(e_1(0), e_2(0))$ of the Lyapunov function (15) and the function

$$\gamma = \min\left\{\frac{2k_1k_3}{4k_3 + 3k_1^2}, \frac{\varepsilon k_1}{2}, \frac{2k_1k_4}{2k_4 + 3(k_1^2 + k_2^2)}\right\} \tag{18}$$

of the system parameters. Recall that the settling-time function

$$\begin{aligned}&T(e_1^0, e_2^0) =\\ &inf\{T \ge 0 : e_1(t, e_1^0) = e_2(t, e_2^0) = 0 \text{ for all } t \ge T\}\end{aligned} \tag{19}$$

is defined for a solution $e_1(t, e_1^0), e_2(t, e_2^0)$ of (11), initialized with $e_1(0) = e_1^0, e_2(0) = e_2^0$. The following result is in force.

Theorem 3: Given positive k_1, k_2, k_3, k_4, and $\varepsilon \in [\frac{1}{2}, 1)$, system (11) is globally finite time stable, and an upper estimate

$$T(e_1^0, e_2^0) \le v_0^{1-\varepsilon}(1-\varepsilon)^{-1}\kappa^{-1} \tag{20}$$

of the settling-time function holds for the initial value v_0 of the Lyapunov function (15), for $\kappa = \gamma(2k_3)^{1-\varepsilon}$, and for γ given by (18).

Proof: While proving Theorem 2, we have established that no sliding modes occur in system (11) provided that the conditions of Theorem 3 hold. Thus, the Lyapunov function (15) is almost always differentiable along the trajectories of the system in question, namely, for all $t \ge 0$ such that $e_1(t) \ne 0$. For those t, the time derivative of the Lyapunov function, computed according to (11), proves to be negative definite. To conclude this, let us set

$$\begin{aligned}W_{\varepsilon}(e_1, s) = [&k_2s^2|e_1|^{1-\varepsilon} + k_1\varepsilon s^2 + k_1k_4e_1^2\\ &+k_2k_3|e_1|^{2-\varepsilon} + k_2k_4|e_1|^{3-\varepsilon} + k_1k_3|e_1|].\end{aligned} \tag{21}$$

Then taking into account that

$$|e_1|^{\varepsilon-1} \ge \left(\frac{V_1}{2k_3}\right)^{\varepsilon-1}, \tag{22}$$

one arrives at

$$\begin{aligned}\dot{V}_1 &= 2sk_3\text{sign } e_1 + 2sk_4e_1 - e_2(k_3\text{sign } e_1 + k_4e_1)\\ &\quad -s[\varepsilon k_1 s|e_1|^{\varepsilon-1} + k_3\text{sign } e_1 + k_4e_1 + k_2s]\\ &= -k_2s^2 - k_1\varepsilon s^2|e_1|^{\varepsilon-1} - k_1k_4|e_1|^{\varepsilon+1}\\ &\quad -k_2k_3|e_1| - k_2k_4e_1^2 - k_1k_3|e_1|^{\varepsilon}\\ &= -|e_1|^{\varepsilon-1}[k_2s^2|e_1|^{1-\varepsilon} + k_1\varepsilon s^2 + k_1k_4e_1^2\\ &\quad +k_2k_3|e_1|^{2-\varepsilon} + k_2k_4|e_1|^{3-\varepsilon} + k_1k_3|e_1|]\\ &\le -\left(\frac{V_1}{2k_3}\right)^{\varepsilon-1} W_\varepsilon(e_1,s) \end{aligned} \tag{23}$$

Employing the well-known inequality $2ab \le a^2+b^2,\ a,b \in R$ and taking into account that under the condition $\varepsilon \in [\frac{1}{2},1)$ of the theorem, $|e_1|^{2\varepsilon} \le |e_1|^2$ for $|e_1| \ge 1$ and $|e_1|^{2\varepsilon} \le |e_1|$ otherwise, the validity of the estimate

$$\begin{aligned} V_1(e_1,e_2) &\le 2k_3|e_1| + (k_4 + \frac{3}{2}k_2^2)e_1^2 + 2s^2 + \frac{3}{2}k_1^2e_1^{2\varepsilon}\\ &\le (2k_3 + \frac{3}{2}k_1^2)|e_1| + (k_4 + \frac{3}{2}k_2^2 + \frac{3}{2}k_1^2)e_1^2 + 2s^2 \end{aligned} \tag{24}$$

is then verified for all $e_1, e_2 \in R$ and s, governed by (16). It is therefore clear that

$$W_\varepsilon(e_1,s) \ge \gamma V_1(e_1,e_2) \tag{25}$$

for γ, governed by (18), and for all $e_1, e_2, s \in R$ such that (16) holds. Thus, inequality (17) is derived from (18), (23) – (25) with $\kappa = \gamma(2k_3)^{1-\varepsilon}$.

To complete the proof it remains to demonstrate that system (11) reaches the origin in finite time and the settling-time function (19) admits estimate (20). For this purpose, let us note that for an admissible $\varepsilon \in [\frac{1}{2},1)$, the solution of the Cauchy problem

$$\dot{v}(t) = -\kappa v^\varepsilon(t), v(0) = v_0 \tag{26}$$

is given by $v(t) = [v_0^{1-\varepsilon} - (1-\varepsilon)\kappa t]^{\frac{1}{1-\varepsilon}}$ and it vanishes at $T = v_0^{1-\varepsilon}(1-\varepsilon)^{-1}\kappa^{-1}$. By applying comparison principle [14], the solutions $V_1(t)$ of the differential inequality (17), initialized with $V_1(0) = v_0$, are upper estimated by the solution $v(t)$ of the Cauchy problem (26), thereby ensuring that the Lyapunov function (15), computed on the solutions of the system in question, is nullified after the same time instant T. The validity of Theorem 3 is thus concluded. ■

In the rest of this section, we carry out relations between the observer gains $k_i,\ i = 1,2,3,4$ that could ensure the robustness of the perturbed dynamics

$$\begin{aligned}\dot{e}_1 &= e_2 - k_1|e_1|^\varepsilon sign\ e_1 - k_2e_1,\\ \dot{e}_2 &= w - k_3 sign\ e_1 - k_4e_1. \end{aligned} \tag{27}$$

As a matter of fact, these dynamics correspond to the observation errors $e = (e_1,e_2)^T,\ e_1 = x - \hat{x},\ e_1 = y - \hat{y}$ between the state of the double integrator (3), affected by an admissible external disturbance (6), and that of the velocity observer (9).

The desired relations, which are found for the admissible disturbances (6) with $\alpha = 1-\varepsilon$ and $\varepsilon \in [\frac{1}{2},\frac{2}{3}]$, depend on the disturbance upper bound μ_0 in the growth condition (6) and they are as follows:

$$\begin{aligned} k_1 > 0,\ k_2 > 1,\ k_3 > \max\left\{\frac{\mu_0k_1}{k_2}, \frac{\mu_0(\mu_0+k_2)}{k_1}\right\},\\ k_4 > \frac{\mu_0(\mu_0+k_2)}{k_2}. \end{aligned} \tag{28}$$

Theorem 4: Consider system (27) under the growth condition (6) on the external disturbance $w(t)$. Let

$$\varepsilon \in [\frac{1}{2},\frac{2}{3}] \quad \text{and} \quad \alpha = 1-\varepsilon, \tag{29}$$

and let the system parameters meet condition (28). Then system (27) is globally finite time stable for any admissible disturbance (6), and the corresponding settling-time function is estimated by (20) with $\kappa = \gamma_1(2k_3)^{1-\varepsilon}$ and γ_1 given by

$$\gamma_1 = \min\left\{\frac{2(k_1k_3 - \mu_0^2 - \mu_0k_2}{4k_3 + 3k_1^2}, \frac{\varepsilon k_1}{2}, \frac{2k_1k_4}{2k_4 + 3(k_1^2+k_2^2)}\right\} \tag{30}$$

Proof: Differentiation (23) of the Lyapunov function (15) along the perturbed system (27) beyond the discontinuity line $e_1 = 0$ is readily modified to

$$\begin{aligned}\dot{V}_1 &= -|e_1|^{\varepsilon-1}W_\varepsilon(e_1,s) + 2sw + k_1w|e_1|^\varepsilon sign\ e_1\\ &\quad +k_2we_1. \end{aligned} \tag{31}$$

By taking into account (25) and substituting the upper estimate (6) subject to (29) for the disturbance w, this yields

$$\begin{aligned}\dot{V}_1 &\le -|e_1|^{\varepsilon-1}W_\varepsilon(e_1,s) + s^2 + w^2\\ &\quad +k_1w|e_1|^\varepsilon sign\ e_1 + k_2we_1\\ &\le -|e_1|^{\varepsilon-1}[W_\varepsilon(e_1,s) - s^2|e_1|^{1-\varepsilon} - w^2|e_1|^{1-\varepsilon}\\ &\quad -k_1we_1 - k_2w|e_1|^{2-\varepsilon}]\\ &\le -|e_1|^{\varepsilon-1}[W_\varepsilon(e_1,s) - s^2|e_1|^{1-\varepsilon} - \mu_0^2|e_1|^{3-3\varepsilon}\\ &\quad -\mu_0k_1|e_1|^{2-\varepsilon} - \mu_0k_2|e_1|^{3-2\varepsilon}] \end{aligned} \tag{32}$$

for all $t \ge 0$ when $e_1(t) \ne 0$. As established in the proof of Theorem 2, no sliding modes occur on the discontinuity line $e_1(t) = 0$. Thus, (29) holds almost for all positive t. Since $|e_1|^{3-i\varepsilon} \le |e_1|^{3-\varepsilon},\ i = 2,3$ for $|e_1| \ge 1$ and $|e_1|^{3-i\varepsilon} \le |e_1|,\ i = 2,3,$ otherwise, it follows that the inequality

$$\dot{V}_1 \le -|e_1|^{\varepsilon-1}W_1(e_1,s) \tag{33}$$

is in force almost for all $t \geq 0$ where

$$\begin{aligned} W_1(e_1,s) \;=\; & [(k_2-1)s^2|e_1|^{1-\varepsilon}+k_1\varepsilon s^2+k_1k_4e_1^2 \\ & +(k_2k_3-\mu_0k_1)|e_1|^{2-\varepsilon} \\ & +(k_1k_3-\mu_0^2-\mu_0k_2)|e_1| \\ & +(k_2k_4-\mu_0^2-\mu_0k_2)|e_1|^{3-\varepsilon}]. \end{aligned} \tag{34}$$

Furthermore, the estimate

$$W_1(e_1,s) \geq \gamma_1 V_1(e_1,e_2) \tag{35}$$

similar to (25), is derived for γ_1, given by (30), and for all $e_1, e_2, s \in R$ such that (16), (28), (29) hold. Relations (22), (32) – (35), coupled together, result in (17) with $\kappa = \gamma_1(2k_3)^{1-\varepsilon}$. The finite time stability of the perturbed system (27) and the corresponding settling-time function estimate are then concluded from inequality (17) by applying the same line of reasoning as that used in the proof of Theorem 3. ■

It is of interest to note that system (27), specified with $\varepsilon = \frac{1}{2}$, remains finite time stable even if affected by a non-vanishing external disturbance. For later use, we denote

$$\gamma_2 = \begin{cases} \min\{\frac{2(k_1k_3-M-Mk_1)}{4k_3+3k_1^2}, \frac{k_1-2M}{4}\} \; if \; k_2=k_4=0 \\ \min\{\frac{2(k_1k_3-M-Mk_1)}{4k_3+3k_1^2}, \frac{k_1-2M}{4}, \frac{2k_1k_4}{2k_4+3k_2^2}\} \text{ otherwise} \end{cases} \tag{36}$$

Theorem 5: Let system (27) be specified with $\varepsilon = \frac{1}{2}$ and let it be affected by a uniformly bounded disturbance (4). Furthermore, let the system gains be such that

$$\begin{aligned} & k_1, k_3 > 0, k_2, k_4 \geq 0, \\ & \text{if } k_4 = 0 \text{ then } k_2 = 0. \end{aligned} \tag{37}$$

Then system (27) is globally finite time stable whenever the upper bound M on the magnitude of the external disturbance w meets the condition

$$M < \min\{\frac{k_1}{2}, \frac{k_1k_3}{1+k_1}\}. \tag{38}$$

In addition, the corresponding settling-time function is estimated by (20) with $\kappa = \gamma_2(2k_3)^{1-\varepsilon}$ and γ_2 given by (36).

Proof: Under the conditions of Theorem 5, differentiation (31) of the Lyapunov function (15) along system (27) beyond the discontinuity line $e_1 = 0$ is specified to

$$\begin{aligned} \dot V_1 \;\leq\; & -|e_1|^{-\frac{1}{2}}W_{\varepsilon=\frac{1}{2}}(e_1,s)+2M|s|+Mk_1|e_1|^{\frac{1}{2}}+Mk_2|e_1| \\ =\; & -|e_1|^{-\frac{1}{2}}\{W_{\varepsilon=\frac{1}{2}}-M[2|s||e_1|^{\frac{1}{2}}-k_1|e_1|-k_2|e_1|^{\frac{3}{2}}]\} \\ & \leq -|e_1|^{-\frac{1}{2}}W_2(e_1,s) \end{aligned} \tag{39}$$

where the well-known inequality $2|s||e_1|^{\frac{1}{2}} \leq s^2+|e_1|$ is taken into account and

$$\begin{aligned} W_2(e_1,s) \;=\; & [k_2s^2|e_1|^{\frac{1}{2}}+(\frac{k_1}{2}-M)s^2+k_1k_4e_1^2 \\ & +(k_2k_3-Mk_2)|e_1|^{\frac{3}{2}} \\ & +(k_1k_3-M-Mk_1)|e_1|k_2k_4|e_1|^{\frac{5}{2}}]. \end{aligned}$$

Since the first inequality of (24) reduces now to

$$V_1(e_1,e_2) \leq (2k_3+\frac{3}{2}k_1^2)|e_1|+(k_4+\frac{3}{2}k_2^2)e_1^2+2s^2, \tag{40}$$

it follows that

$$W_2(e_1,s) \geq \gamma_2 V_1(e_1,e_2) \tag{41}$$

for γ_2, given by (36), and for all $e_1, e_2, s \in R$ such that relations (16), (37), (38) hold. Similar to that of the proof of Theorem 4, relation (17) with $\varepsilon = \frac{1}{2}$ and $\kappa = \gamma_2(2k_3)^{1-\varepsilon}$ is then derived from (22), (39) – (41), coupled together. This completes the proof of Theorem 5 because as shown in the proof of Theorem 3, inequality (17) ensures that the Lyapunov function (15) is nullified after the finite time instant $T = v_0^{1-\varepsilon}(1-\varepsilon)^{-1}\kappa^{-1}$, thereby yielding both the global finite time stability of the system in question and the corresponding settling-time function estimate. ■

IV. Finite Time Stabilizing Output Feedback Synthesis

In this section, we proceed with the design of the output feedback, stabilizing the double integrator in finite time. For this purpose, we substitute the velocity y in the state feedback (2) by its estimate $\hat y$ to arrive at the finite time stabilizing output feedback

$$u = -\mu|\hat y|^{\alpha} sign\ \hat y - \nu|x|^{\frac{\alpha}{2-\alpha}} sign\ x, \tag{42}$$

so that the resulting closed-loop system proves to be globally finite time stable regardless of whichever admissible disturbance affects the system.

Theorem 6: Consider system (3) under the growth condition (6) on the external disturbance w. Let (3) be driven by the observer-based dynamic feedback (9), (42) with parameters α, ε subject to (29), with positive controller gains μ, ν such that $\mu > \mu_0$, and with observer parameters $k_i,\ i = 1,2,3,4$, satisfying condition (28). Then the closed-loop system (3), (9), (42) is globally asymptotically stable, regardless of whichever external disturbance (6) affects the system.

Proof: Clearly, the closed-loop system (3), (9), (42), rewritten in terms of the observation error (27), meet the conditions of Theorem 4. By applying Theorem 4 to the observation error system (27), it is concluded that starting from a finite time instant T, the closed-loop system evolves on the manifold $e_1 = e_2 = 0$ where $\hat y = y$, thereby ensuring that the output control signal (42) coincides with the state feedback signal (2). Due to this, the linear growth conditions $|w(x, y, t)| \le \mu_0(1+|y|)$ and $|u(x, \hat{y})| \le \mu(1+|\hat{y}|)+\nu(1+|x|)$ turn out to hold for all $x, y \in R,\ t \ge 0$ and for the external disturbance (6) and the control input (42), respectively, so that the closed-loop state is unlimitedly extendible to the right. To complete the proof it remains to apply Theorem 1 to (3), (9), (42) for $t \ge T$ when the output feedback equals the state feedback. The global asymptotic stability of the closed-loop system (3), (9), (42) is thus established. ∎

To this end, we present the output feedback controller that additionally imposes on the closed-loop system global finite time stability and robustness against uniformly bounded disturbances.

Theorem 7: Let system (3) be affected by a uniformly bounded disturbance (4) and let it be driven by the observer-based dynamic feedback (9), (42) specified with $\alpha = 0$ and $\varepsilon = \frac{1}{2}$, with positive controller gains μ, ν subject to (5), and with observer parameters $k_i,\ i = 1, 2, 3, 4$, satisfying conditions (37), (38). Then the closed-loop system (3), (9), (42) is globally finite time stable.

Proof: By applying Theorem 5 to the observation error system (27), it is concluded that starting from a finite time instant T, the closed-loop system evolves on the manifold $e_1 = e_2 = 0$ where $\hat{y} = y$, thereby ensuring that the output control signal (42) coincides with the state feedback signal (2). Since the linear growth conditions $|w(x, y, t)| \le M$ and $|u(x, \hat{y})| \le \mu(1+|\hat{y}|)+\nu(1+|x|)$ turn out to hold for all $x, y \in R,\ t \ge 0$ and for the external disturbance (4) and the control input (42), respectively, the closed-loop state is unlimitedly extendible to the right. Now applying [4, Theorem 4.2] to (3), (9), (42), provided that for $t \ge T$, the output feedback equals the state feedback, yields the global finite time stability of the closed-loop system (3), (9), (42). The proof is thus completed. ∎

Numerical simulations, made for the double integrator (3), driven by the observer-based dynamic feedback (9), (42), are presented in Fig.1. Good performance of the developed synthesis and its strong robustness feature against the applied sinusoidal disturbance $w = 0.5sin100t$ are concluded from this figure.

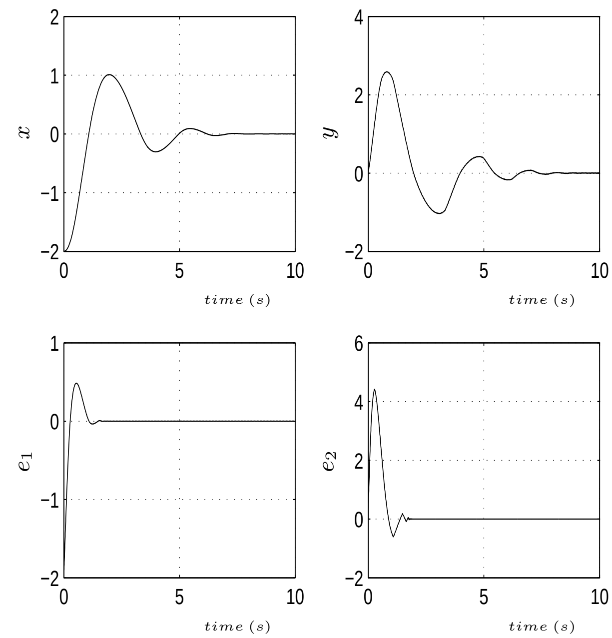

Fig. 1. Output feedback stabilization (9), (42) of the perturbed double integrator (3), initialized with $x(0) = -2,\ y(0) = 0$ and specified with $\mu = 1, \nu = 2,\ k_1 = k_3 = 1,\ k_2 = k_4 = 2,\ \alpha = \frac{1}{2}$, and $\varepsilon = \frac{1}{2}$. The case where the external sinusoidal perturbation $w = 0.5sin100t$ is applied.

V. Conclusions

A modification of the twisting controller and that of the supertwisting observer are coupled together to present a unified framework for the finite time output feedback stabilization of a double integrator. The proposed modifications are shown to inherit, from their originators, desired finite time stability and robustness properties. These features make the proposed synthesis attractive for further extension to electromechanical applications with hard-to-model nonlinear phenomena such as friction and impacts. Finite time orbital stabilization of a biped robot is among open problems to be tackled through the developed approach. This work is in progress and it is going to be published in [15].

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