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Improved active learning probabilistic approach for the computation of failure probability

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ABSTRACT

This paper presents a cost-effective probabilistic approach to be used in engineering applications. The proposed approach consists of an improved Kriging-based method aiming at reducing to a minimum the number of evaluations of the true performance function when computing a failure probability. It is a kind of variant of the classical Active learning method combining Kriging and Monte Carlo Simulation (AK-MCS) developed by Echard et al. (2011) [1], where some improvements are introduced to enhance the learning process. Some illustrative and practical examples are presented and discussed. The proposed approach has shown a great efficiency as compared to the classical AK-MCS approach.

1. Introduction

The conventional method for the assessment of failure probability of an engineering system is the crude Monte Carlo Simulation (MCS). This method is considered as a reference tool when performing a probabilistic analysis due to its accuracy and ease of implementation. The estimation of the failure probability by MCS methodology requires the evaluation of the performance function for the whole Monte Carlo population. This might be an easy task when the performance function is expressed by an analytical equation for which the computational cost is negligible, but it becomes a great impediment when computationally-expensive computer codes such as finite element/finite difference models are used. Therefore, it is desirable to develop alternative more efficient probabilistic approaches to determine the failure probability estimate with a minimum number of calls to the computationally-expensive computer code.

FORM and SORM approximate methods [2] may be very efficient for the computation of the failure probability due to the fact that only a relatively small number of model evaluations is needed to find the Most Probable Failure Point (MPFP). Notice however that the use of these methods may not be acceptable in many practical problems involving a high stochastic dimension or a nonlinear limit state surface.

Metamodeling techniques are often used in the literature in the domains of design optimization and reliability analysis because of the significant number of simulations that are required within these domains. The metamodeling techniques aim at approximating the model response by a surrogate model (called also metamodel). Various types of metamodeling techniques can be found in the literature such as the Response Surface Methodology (RSM) [3–5], the Polynomial Chaos Expansion (PCE) and its extension the Sparse Polynomial Chaos Expansion (SPCE) [6–13], the Artificial Neural Networks (ANN) [14], the Support Vector Machine (SVM) [15–18] and the Kriging method [19–23]. Notice that each type of metamodel is characterized by its own underlying assumptions and that the approximation of the model responses depends on the type of the selected surrogate model [24].

In the past few decades, interesting probabilistic approaches based on metamodeling techniques and aiming at alleviating the computational cost of the simulation methods (i.e. the crude MCS and the variance reduction techniques as Importance Sampling IS and Subset Simulation SS) have gained a lot of interest. The principle of these approaches may be summarized in two main steps: (i) substituting the system model which may consist in a costly-to-evaluate finite element/finite difference model with a cheap-to-evaluate surrogate model that should be sufficiently accurate and (ii) performing the probabilistic analysis (i.e. computing the failure probability) using one of the simulation methods on the basis of the obtained time-efficient surrogate model. These approaches may be defined as adaptive approaches [24]. This means that an initial set of few points (called initial Design of experiments DoE) is firstly selected to be computed by the true performance function and to be used as input for the construction of a preliminary surrogate model and secondly, the constructed surrogate model is iteratively updated (by updating the DOE with additional training points) *via* an active learning process until sufficient accuracy is achieved. In this regards, various advanced probabilistic approaches

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combining a metamodeling technique with a simulation method have been reported in the literature. For instance, [25] proposed a method combining subset simulation and support vector machines, [26] suggested a combination between subset simulation and Kriging, and [1,27] combined Kriging metamodeling with Monte Carlo and Importance Sampling leading to what is called AK-MCS and AK-IS approaches respectively. All these methods take advantage of both the metamodeling and the simulation techniques. They aim at efficiently computing a small failure probability based on the constructed metamodel making use of a reduced number of calls to the true performance function.

Among the different metamodeling techniques, the Kriging metamodeling has gained attention in the domain of reliability analysis since it presents several interesting features compared to the other types of metamodeling. Kriging is an exact interpolation method that provides (thanks to its stochastic property) not only the predicted value at a certain point, but also the estimation of the local variance of this prediction that defines the local uncertainty on the prediction. It was shown to be a relevant tool for the efficient assessment of failure probability due to its flexibility and adaptation to a wide range of model responses.

Within the Kriging-based probabilistic methods, the AK-MCS approach by [1], which is an Active learning method that combines Kriging metamodeling and Monte Carlo Simulation, has gained popularity in the literature. This method involves the construction of an approximate Kriging metamodel on the basis of the responses of a small DoE computed using the system model. This approximate Kriging metamodel is then successively updated *via* an enrichment process making use of a learning function that takes benefits from the Kriging characteristics. Once the stopping criterion indicates that the Kriging metamodel is sufficiently improved, MCS methodology is applied on the obtained Kriging surrogate model instead of the system model in order to estimate the probability of failure.

In AK-MCS method, the training point chosen for the enrichment process is selected as the one having the highest probability of being misclassified among all the candidate points. Notice however that the chosen point does not reduce most efficiently the variance of P_f because one does not take into account the correlations between the points predictions as will be shown later in this paper. Moreover, the enrichment in AK-MCS approach stops when satisfying a criterion that ensures an appropriate classification (safe/failure) of the whole MC population points. Such a stopping criterion is not very relevant because it is not based on the failure probability estimate. A variant of AK-MCS approach is proposed in this paper. It is called herein AK-MCSd where d stands for dependent Kriging predictions. Within this approach, one takes benefits of the dependencies between the Kriging predictions. This is done by using the complete Gaussian process output of the Kriging metamodel as was suggested by [28]. In other words, not only the mean predictions and the corresponding prediction variances are to be considered, but also the correlations between the Kriging predictions at all candidate points shall be taken into account in the enrichment process. Concerning the stopping criterion of AK-MCSd approach, a relevant stopping criterion that is based on the measure of the gap between the estimated value of the failure probability and the expectation of this failure probability (that takes into account the uncertainty of the Kriging predictions) is suggested. This criterion allows one to focus on the accuracy of the reliability estimate (i.e. the failure probability) instead of focusing on an appropriate classification of the whole MC population points as suggested in AK-MCS approach. This may provide more efficiency to the proposed AK-MCSd approach by avoiding excessive computations of unnecessary extra training points responses.

This paper is organized as follows: The next two sections present an overview on the AK-based methods available in literature and the Kriging metamodeling theory used in the present probabilistic approach. This is followed by a brief description of the classical Kriging-based AK-MCS approach. Then, the proposed AK-MCSd probabilistic approach is presented in some detail. Finally, the proposed approach is applied to some illustrative and practical problems.

2. Overview on the AK-based methods

Although AK-MCS method has shown great efficiency in many cases, it nevertheless has several weaknesses. One may cite (i) the large population that is required when estimating very small values of the failure probability, (ii) the low efficiency of the method when dealing with system reliability problems, (iii) the point-by-point enrichment process that is used for learning, (iv) the type of the adopted learning function used for the selection of the training points and (v) the stopping condition for learning that is often agreed to be too conservative.

To solve the large population problem, [27,29] replaced the original population with the population generated by IS and SS respectively. An extension of the method by [27] was proposed by [30] to deal with problems involving multiple failure regions. Xu et al. [31] proposed a new approach called AK-MSS that combines AK-MCS and the modified subset simulation. This approach replaces the large population with a population that consists of conditional samples that are generated by the MSS. Moreover, Lelièvre et al. [32] made use of a sequential MCS technique to estimate the small failure probabilities. Other adaptive methods based on Kriging have been developed for system reliability problems by [33–35]. In order to overcome the point-by-point enrichment process that is used for learning in AK-MCS, [32] proposed a multi-point enrichment process based on an improved clustering technique.

Concerning the use of learning functions for the selection of training points, the expected feasibility function (EFF) proposed by [23] and the U function developed by [1,27] select points near the limit state surface of the Kriging model. Yang et al. [36] suggested identifying the new training point by selecting the point having the maximum value of the expected risk function (ERF). This corresponds to the point for which the sign of the response has the largest risk to be wrongly predicted. Lv et al. [37] proposed the use of the information entropy function H where the new training point is selected such that it has the maximal value of H. Indeed, the prediction is more certain when the information entropy is lower. Sun et al. [38] developed the least improvement function (LIF) which quantifies how much the accuracy of the failure probability estimate will be improved when a new point is added to the DoE. Finally, Zhang et al. [39] proposed a novel active learning function called Reliability-based expected improvement function (REIF).

Concerning the stopping criterion for learning, many authors realized that the U-criterion adopted by AK-MCS may be too conservative for engineering applications. [35] stated that the accuracy of the Kriging model may be considered as acceptable if fewer than 2% of the points violate the stopping criterion defined in AK-MCS. Gaspar et al. [40,41] proposed an additional convergence criterion to the one proposed in AK-MCS that exploits the stabilization of the failure probability estimate during the active learning process in order to provide a compromise between the accuracy of the Kriging metamodel and the computation cost. Wang et al. [42] defined a cumulative confidence level CCL measure of the Kriging model to quantify the accuracy of the reliability estimate and considered the metamodel as acceptable when its CCL is above a given confidence target. Schöbi et al. [43] defined a limit state margin characterized by upper and lower boundaries of the limit state surface that takes into account the prediction uncertainty in the Kriging metamodel. These authors stated that when these boundaries become close to each other, a thin limit state margin is obtained and thus, the estimated failure probability can be considered as accurate. Finally, Jian et al. [44] defined two accuracy measures that determine how well the Kriging metamodel and the estimate of failure probability are. As mentioned by these authors, the two measures may be used to construct a stopping criterion within a reliability analysis.

Other improvement within the AK-based methods was proposed by

Gaspar et al. [41] and Cheng and Lu [45]. [41] proposed an adaptive surrogate model with active refinement combining Kriging and a trust region method. [45] developed a new adaptive approach for reliability analysis *via* an ensemble learning of multiple competitive surrogate models, including Kriging, polynomial chaos expansion and support vector regression. The developed approach consists in fitting the performance function with multiple different surrogate models to get a more robust approximation of this function through a weighted average strategy.

3. Kriging metamodeling theory

The Kriging metamodeling assumes that the performance function $G(\boldsymbol{x})$ (where $\boldsymbol{x}$ is a n-dimensional vector, n being the number of random variables) is a realization of a Gaussian process $\mathscr{G}(\boldsymbol{x})$ that is composed of a deterministic trend $F(\boldsymbol{x}, \boldsymbol{\beta})$ and a centered stochastic process $Z(\boldsymbol{x})$. It can be described by the following equation [19]:

$$\mathscr{G}(\boldsymbol{x}) = F(\boldsymbol{x}, \boldsymbol{\beta}) + Z(\boldsymbol{x}) \tag{1}$$

where the deterministic part $F(\boldsymbol{x}, \boldsymbol{\beta})$ corresponds to a regression model that can be written as follows:

$$F(\boldsymbol{x}, \boldsymbol{\beta}) = \boldsymbol{f}(\boldsymbol{x})^T\boldsymbol{\beta} \tag{2}$$

In this equation, $\boldsymbol{f}(\boldsymbol{x})^T = [f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \cdots, f_p(\boldsymbol{x})]$ is a vector of regression functions and $\boldsymbol{\beta}^T = [\beta_1, \beta_2, \cdots, \beta_p]$ is a vector of regression coefficients. The stochastic process $Z(\boldsymbol{x})$ represents the fluctuations around the mean trend $F(\boldsymbol{x}, \boldsymbol{\beta})$. It interpolates the gaps between the regression model and the true performance function values at the different N points of the DoE. It is defined by a stationary Gaussian process with zero mean and covariance given as follows:

$$\text{cov}(\boldsymbol{x}, \boldsymbol{x}') = \sigma_z^2\ R(\boldsymbol{x}, \boldsymbol{x}') \tag{3}$$

where σ_z^2 is the process variance and R is the correlation function between two arbitrary points $\boldsymbol{x}$ and $\boldsymbol{x}$' of the DoE. This function is defined by its corresponding set of correlation parameters θ, where θ is a vector of dimension n. Several models exist to define the correlation function, the most commonly used being the anisotropic square-exponential function (or the anisotropic Gaussian function) given as follows:

$$R(\boldsymbol{x}, \boldsymbol{x}') = \prod_{k=1}^{n} e^{(-\theta_k (x_k - x_k')^2)} \tag{4}$$

where x_k and x'_k are the k^{th} coordinates of the points $\boldsymbol{x}$ and $\boldsymbol{x}$' and θ_k is a scalar which is equal to the inverse of the correlation length in the k^{th} direction. There are different types of Kriging: (i) Simple Kriging that assumes a known constant trend, (ii) Ordinary Kriging that assumes an unknown constant trend and (iii) Universal Kriging that assumes a general polynomial trend model. In this paper, ordinary Kriging is used and thus, $F(\boldsymbol{x}, \boldsymbol{\beta})$ is replaced by a scalar β to be determined. Notice that all the following equations are based on ordinary Kriging assumption.

In order to predict the value of the performance function $G(\boldsymbol{x})$ at an unknown point $\boldsymbol{x}$, the Best Linear Unbiased Predictor BLUP $G_p(\boldsymbol{x})$ of $\mathscr{G}(\boldsymbol{x})$ is shown to be a Gaussian random variate characterized by a mean prediction value $\mu_{G_p(\boldsymbol{x})}$ and a corresponding prediction variance $\sigma^2_{G_p(\boldsymbol{x})}$ as follows:

$$\mu_{G_p(\boldsymbol{x})} = \beta + \boldsymbol{r}(\boldsymbol{x})^T\boldsymbol{R}^{-1}(\boldsymbol{G} - \beta\boldsymbol{F}) \tag{5}$$

$$\sigma^2_{G_p(\boldsymbol{x})} = \sigma_z^2 \left\{ \begin{array}{c} 1 - \boldsymbol{r}(\boldsymbol{x})^T\boldsymbol{R}^{-1}\boldsymbol{r}(\boldsymbol{x}) \\ + [\boldsymbol{F}^T\boldsymbol{R}^{-1}\boldsymbol{r}(\boldsymbol{x}) - 1]^T(\boldsymbol{F}^T\boldsymbol{R}^{-1}\boldsymbol{F})^{-1} \\ [\boldsymbol{F}^T\boldsymbol{R}^{-1}\boldsymbol{r}(\boldsymbol{x}) - 1] \end{array} \right\} \tag{6}$$

where $\boldsymbol{G} = [G(\boldsymbol{x}^1), G(\boldsymbol{x}^2), \cdots, G(\boldsymbol{x}^N)]^T$ is a vector of exact responses at the training points (i.e. the points of the DoE), $\boldsymbol{r}(\boldsymbol{x}) = [R(\boldsymbol{x}, \boldsymbol{x}^1), R(\boldsymbol{x}, \boldsymbol{x}^2), \cdots, R(\boldsymbol{x}, \boldsymbol{x}^N)]^T$ is a correlation vector containing the correlation between the point $\boldsymbol{x}$ and each of the N training points, $\boldsymbol{R} = [R(\boldsymbol{x}^{(i)}, \boldsymbol{x}^{(j)})](i, j = 1, 2, \cdots, N)$ is a correlation matrix containing the values of the correlation function for all possible combinations of the N training points and $\boldsymbol{F}$ is a vector of length N filled with 1. Notice also that the prediction responses at two given points $\boldsymbol{x}$ and $\boldsymbol{x}$' are correlated random variates having the following covariance:

$$\text{covar}(\boldsymbol{x}, \boldsymbol{x}') = \sigma_z^2 \left\{ \begin{array}{c} R(\boldsymbol{x}, \boldsymbol{x}') - \boldsymbol{r}(\boldsymbol{x})^T\ \boldsymbol{R}^{-1}\boldsymbol{r}(\boldsymbol{x}') \\ + [1 - \boldsymbol{F}^T\boldsymbol{R}^{-1}\boldsymbol{r}(\boldsymbol{x})]^T(\boldsymbol{F}^T\boldsymbol{R}^{-1}\boldsymbol{F})^{-1} \\ [1 - \boldsymbol{F}^T\boldsymbol{R}^{-1}\boldsymbol{r}(\boldsymbol{x}')] \end{array} \right\} \tag{7}$$

According to [46], the scalar β and the process variance σ_z^2 may be estimated by:

$$\hat{\beta} = (\boldsymbol{F}^T\boldsymbol{R}^{-1}\boldsymbol{F})^{-1}\boldsymbol{F}^T\boldsymbol{R}^{-1}\boldsymbol{G} \tag{8}$$

$$\hat{\sigma}_z^2 = \frac{(\boldsymbol{G} - \beta\boldsymbol{F})^T\boldsymbol{R}^{-1}(\boldsymbol{G} - \beta\boldsymbol{F})}{N} \tag{9}$$

Both $\hat{\beta}$ and $\hat{\sigma}_z^2$ depend on the set of correlation parameters θ through the matrix $\boldsymbol{R}$. These parameters can be obtained by solving an optimization problem making use of the maximum likelihood estimation MLE method.

Notice that the construction of a Kriging metamodel (i.e. the determination of the set of correlation parameters θ, the scalar β and the process variance σ_z^2) and the computation of Kriging predictions at unknown points (i.e. at points outside the DoE) can be easily performed using DACE (Design and Analysis of Computer Experiments) toolbox in Matlab. For more details, the reader may refer to [47]. Notice here that the variances of the training points (i.e. the points of the DoE) used for the construction of the metamodel are zero, i.e. the corresponding predictions are exact. However, the variances of the other points (i.e. the points outside the DoE) are always different from zero and they are as large as the corresponding predictions are not accurate. The prediction variance was used by [1] as a key parameter for the learning of the Kriging metamodel when performing a Kriging-based probabilistic analysis.

4. Active learning method combining Kriging and Monte Carlo Simulation (AK-MCS)

The Active learning method combining Kriging and Monte Carlo Simulation (named AK-MCS method) was developed by [1]. It involves the two main stages:

Construction of a preliminary Kriging metamodel

1. Generation of a large Monte Carlo population S in the design space. This population is composed of N_{MCS} points $\boldsymbol{x}^{(i)}$ (i = 1, ...,N_{MCS}).
2. An initial small DoE is randomly selected among the population S. These points are evaluated on the real performance function and are used to construct a preliminary Kriging metamodel.

Enrichment process

1. The Kriging predictions $\mu_{G_p(\boldsymbol{x})}$ and their corresponding Kriging predictions variances $\sigma^2_{G_p(\boldsymbol{x})}$ are computed for the whole population S according to Eqs. (5) and (6). The probability of failure is then estimated as follows:

$$P_f = \frac{1}{N_{MCS}} \sum_{i=1}^{N_{MCS}} I(G_p(\boldsymbol{x}^{(i)})) \approx \frac{N_{\mu_{G_p(\boldsymbol{x})} \le 0}}{N_{MCS}} \tag{10}$$

where $G_p(\boldsymbol{x}^{(i)})$ in this equation is the metamodel random response at the point $\boldsymbol{x}^{(i)}$, N_{MCS} is the number of MCS points (taken here equal to 5×10^5 points) and I is the indicator function such that $I(G_p(\boldsymbol{x}^{(i)})) = 1$ if $G_p(\boldsymbol{x}^{(i)}) \le 0$; otherwise, $I(G_p(\boldsymbol{x}^{(i)})) = 0$. Notice that the failure probability P_f in Eq. (10) is computed by replacing the metamodel random responses $G_p(\boldsymbol{x}^{(i)})$, i = 1, 2, $\cdots, N_{MCS}$ by the mean prediction values $\mu_{G_p(\boldsymbol{x}^{(i)})}$ of the Kriging metamodel. In other words, it is obtained as the ratio of the points in the population S with a negative or null Kriging prediction and the total number of points in S.

2. The best next point in S is identified to be evaluated on the real performance function: This is performed by evaluating a learning function $U(\boldsymbol{x}^{(i)})$ for each point in the population S where $U(\boldsymbol{x}^{(i)})$ is given by:

$$U(\boldsymbol{x}^{(i)}) = \frac{|\mu_{G_p(\boldsymbol{x}^{(i)})}|}{\sigma_{G_p(\boldsymbol{x}^{(i)})}} \qquad i = 1, \cdots, N_{MCS} \tag{11}$$

The best next point x^* is chosen as the one with the minimum value of U. This corresponds to the point having the maximal probability of having a wrong performance function sign.

3. The learning stops if the minimum value of U is greater than 2. This corresponds to a probability of a wrong sign of the performance function that is lower than 0.0228 (see [1]). If not, the best point $\boldsymbol{x}^*$ is evaluated on the true performance function and the DoE is updated with this additional point. Then, a new Kriging metamodel is constructed on the basis of the updated DoE and the method goes back to the first step of the enrichment process. This process of learning is repeated until the stopping condition is satisfied.
4. The coefficient of variation of the failure probability estimate is calculated based on the following equation using the final Kriging metamodel:

$$COV(P_f) = \sqrt{\frac{1 - P_f}{P_f . N_{MCS}}} \tag{12}$$

5. Proposed probabilistic approach

The proposed AK-MCSd probabilistic approach aims at improving the performance of AK-MCS approach. It consists of an active learning method combining Kriging and MCS and making use of the Kriging predictions dependencies. In the proposed approach, the failure probability estimate P_f given by Eq. (10) is considered as a random variable because it is a function of Kriging predicted responses $G_p(\boldsymbol{x}^{(i)})$ that are random variates according to the property of the Kriging metamodeling. [28] proposed two formulas for the mean (or expectation) and the variance of the estimated failure probability that involve not only the mean predictions and the corresponding predictions variances, but also the dependencies (i.e. the correlations) between the Kriging predictions. The expectation of the failure probability estimate is given by the following formula:

$$E(P_f) = \frac{1}{N_{MCS}} \sum_{i=1}^{N_{MCS}} e_i \tag{13}$$

where the term e_i is given as follows:

$$e_i = \Phi\left(-\frac{\mu_{G_p(\boldsymbol{x}^{(i)})}}{\sigma_{G_p(\boldsymbol{x}^{(i)})}}\right) \tag{14}$$

In this formula, Φ stands for the cumulative density function CDF of the standard Gaussian distribution. Thus, e_i may be defined as the CDF of the Normal distribution $N(\mu_{G_p(\boldsymbol{x}^{(i)})}, \sigma_{G_p(\boldsymbol{x}^{(i)})})$ at 0 where $\mu_{G_p(\boldsymbol{x}^{(i)})}$ and $\sigma^2_{G_p(\boldsymbol{x}^{(i)})}$ are respectively the mean prediction and prediction variance at the point $\boldsymbol{x}^{(i)}$.

The variance of the failure probability estimate (which represents the error of the failure probability estimate) is given as follows:

$$Var(P_f) = \frac{1}{{N_{MCS}}^2} \sum_{i=1}^{N_{MCS}} c_i \tag{15}$$

where c_i is the contribution of the point $\boldsymbol{x}^{(i)}$ $(i = 1, 2, \cdots, N_{MCS})$ in the variance of the failure probability estimate. It is given as follows [28]:

$$c_i = e_i(1 - e_i) + \sum_{j=1, j\neq i}^{N_{MCS}} (e_{ij} - e_i e_j) \tag{16}$$

Equation (16) can be presented as the summation of two parts: The first part $e_i(1 - e_i)$ represents the individual contribution of each point (independently from the other points) in the uncertainty of the failure probability. The second part $\sum_{j=1,j\neq i}^{N_{MCS}} (e_{ij} - e_i e_j)$ represents the contribution of the mutual effects between points to the uncertainty of the failure probability, where e_{ij} is the joint CDF of the bivariate Normal distribution $N_2(\boldsymbol{\mu}_{ij}, \Sigma_{ij})$ at (0, 0). The term $\boldsymbol{\mu}_{ij}$ in this distribution is a vector containing the mean predictions values $\mu_{G_p(\boldsymbol{x}^{(i)})}$ and $\mu_{G_p(\boldsymbol{x}^{(j)})}$ of the two points $\boldsymbol{x}^{(i)}$ and $\boldsymbol{x}^{(j)}$ respectively and the term Σ_{ij} is the covariance matrix given as follows:

$$\Sigma_{ij} = \begin{bmatrix} \sigma^2_{G_p(\boldsymbol{x}^{(i)})} & covar(\boldsymbol{x}^{(i)}, \boldsymbol{x}^{(j)}) \\ covar(\boldsymbol{x}^{(j)}, \boldsymbol{x}^{(i)}) & \sigma^2_{G_p(\boldsymbol{x}^{(j)})} \end{bmatrix} \tag{17}$$

where $covar(\boldsymbol{x}^{(i)}, \boldsymbol{x}^{(j)}) = covar(\boldsymbol{x}^{(j)}, \boldsymbol{x}^{(i)})$ is the covariance between the predictions at the two points $\boldsymbol{x}^{(i)}$ and $\boldsymbol{x}^{(j)}$. Its expression is given by Eq. (7).

A simple way of learning may consist in choosing among the MCS population the point having the highest contribution c_i in $Var(P_f)$ and stopping the enrichment when $Var(P_f)$ becomes smaller than a certain prescribed threshold. Notice however that the computation of c_i [and thus $Var(P_f)$] is cumbersome because the total number of calculations of the bivariate probabilities e_{ij} $(i = 1, 2, \cdots, N_{MCS};\ j = 1, 2, \cdots, N_{MCS})$ corresponding to the whole population involving N_{MCS} points is very high and thus it requires a large computation time. In order to address this issue, [28] proposed the construction of a subdomain of candidate points where the computation of c_i and $Var(P_f)$ will be based on only this subdomain. In their approach, [28] established a relationship between $Var(P_f)$ and $E(P_f)$ computed on the basis of the subdomain and those computed on the basis of the whole MC population.

To accurately determine $Var(P_f)$ and $E(P_f)$ based on the subdomain, the constructed subdomain has to include all the points in the failure region ($G < 0$) in order to deal with high values of the failure probability and thus small values of the error on this failure probability. The remaining points of the subdomain are chosen such that they have the highest individual contributions $e_i(1 - e_i)$ (first term of Eq. (16)) in the uncertainty of the failure probability among the safe ($G > 0$) MC population points.

It should be emphasized that the subdomain proposed by [28] considers a large amount of points in the failure domain. These points may not be necessarily close to the LSS and thus, they will not efficiently contribute to the improvement of the LSS. A more efficient candidate pool is proposed in this paper where a quite simple candidate selection procedure was adopted. This procedure aims at considering a maximal number of candidate points in the vicinity of the limit state surface. In addition, a new stopping criterion based on the quantity of interest (i.e. failure probability) without resorting to $Var(P_f)$ is proposed.

5.1. Steps of construction of the Kriging metamodel

Two stages are necessary for the construction of the Kriging metamodel in the present AK-MCSd probabilistic approach. The first stage which involves the construction of an approximate preliminary Kriging metamodel based on a small DoE remains similar to that presented in the AK-MCS procedure described above. It will not be repeated in this section. The second stage related to the enrichment process may be described as follows:

The basic idea of the learning process (or enrichment process) within AK-MCSd approach consists in identifying the point having the biggest contribution to the uncertainty in the estimated failure probability. The uncertainty in the estimated failure probability may be

computed by using the formula of the variance of P_f provided by Eq. (15). The identified point is added to the DoE in order to reduce the uncertainty of P_f in the most efficient manner. The step-by-step procedure of the enrichment process may be summarized as follows:

1. Compute for the whole Monte Carlo population (i.e. for all the N_{MCS} points) only the individual contributions to the variance of P_f (i.e. the first part of Eq. (16)) and choose among the whole population only n_{can} points (n_{can} is taken equal to 20 herein) having the highest individual contributions $e_i(1-e_i)$.
2. Compute for the selected n_{can} points the whole formula (Eq. (16)) including both the individual and the mutual contributions. It should be noted that the first step (related to the use of only the first part of Eq. (16)) was adopted in the analysis because the total number of calculations of the bivariate probabilities e_{ij} ($i = 1, 2, \cdots, N_{MCS}$; $j = 1, 2, \cdots, N_{MCS}$) corresponding to the whole population involving N_{MCS} points is equal to $N_{MCS}(N_{MCS}+1)/2$. A such number of computations requires a large computation time. Hence, in order to avoid computing bivariate probabilities e_{ij} for the whole points of the population and thus reducing the corresponding computation time, a reduced number n_{can} of points (instead of N_{MCS} points) was adopted for the computation of the mutual contributions.
3. Select among the n_{can} points, the point having the highest contribution c_i to the uncertainty in the failure probability. Then, compute the response of the selected point by the system model.
4. Add the selected point and the corresponding computed response to the DoE and update the Kriging metamodel.
5. Repeat steps 1–4 until reaching the stopping condition described in the following section.

5.2. Stopping condition

The enrichment process of the Kriging metamodel ends when satisfying a stopping condition. A new stopping criterion is proposed in this paper. It is based on the measure of the gap between the expectation of the failure probability $E(P_f)$ given by Eq. (13) and the estimated value of the failure probability P_f given by Eq. (10). It may be expressed by the following percentage error:

$$\varepsilon_r = \frac{|P_f - E(P_f)|}{P_f} \times 100\% \tag{18}$$

It should be noted that the indicator function I that appears in Eq. (10) presents some uncertainty due to the uncertainty in the predicted random responses $G_p(\boldsymbol{x}^{(i)})$. This uncertainty is not considered in Eq. (10); however, it is taken into account through Eq. (13).

In order to better understand the proposed stopping condition, let us consider the case of a point that has an "**exact**" predicted response and a corresponding standard deviation equal to zero (i.e. a point that does not present any uncertainty in the corresponding predicted response and thus in the indicator function I). This corresponds to a typical point of the DoE. In such case, the value of e_i will be equal to:

$$e_i = \Phi\left(-\frac{\mu_{G_p(\boldsymbol{x}^{(i)})}}{\sigma_{G_p(\boldsymbol{x}^{(i)})}}\right) = \begin{cases} 0, \ if \mu_{G_p(\boldsymbol{x}^{(i)})} > 0 & since \Phi(-\infty) = 0 \\ 1, \ if \mu_{G_p(\boldsymbol{x}^{(i)})} < 0 & since \Phi(+\infty) = 1 \end{cases} \tag{19}$$

Thus, for the case where all points have exact responses [which will be equivalent to the case of application of the crude MCS approach using the system model (not the metamodel) responses], e_i will be equal to $I(G_p(\boldsymbol{x}^{(i)}))$ for all points $\boldsymbol{x}^{(i)}$ $i = 1, 2, \cdots, N_{MCS}$ where $G_p(\boldsymbol{x}^{(i)})$ are herein the exact values of the performance function. For that particular case, Eq. (13) will be identical to Eq. (10). Notice however that the predicted responses of points that are outside the DoE are never completely exact. Indeed, these points present some errors corresponding to the standard deviation values $\sigma_{G_p(\boldsymbol{x}^{(i)})}$ that are different from zero. The more the standard deviation is smaller (i.e. the predicted response is more accurate), the more the value of e_i will be closer to 0 (respectively to 1) if $\mu_{G_p(\boldsymbol{x}^{(i)})} > 0$ (respectively $\mu_{G_p(\boldsymbol{x}^{(i)})} < 0$).

The convergence of the estimated failure probability P_f was thus considered to be achieved when Eqs. (10) and (13) lead to sufficiently close values, i.e. when the error ε_r (given by Eq. (18)) becomes smaller than a prescribed threshold. A threshold value of $\varepsilon_r = 1\%$ was adopted in this work. Compared to the U-criterion adopted within AK-MCS, the present stopping criterion allows one to focus on the accuracy of the reliability estimate (i.e. the failure probability) instead of focusing on the well classification of each point of the MC population.

6. Application examples

In this section, the performance of the proposed AK-MCSd probabilistic approach is firstly checked through two illustrative examples. Secondly, a comparative study with the work of Zhu and Du [28] was performed *via* a third application example. Finally, a practical geotechnical problem involving the study of a monopile foundation embedded in a spatially varying clayey soil was presented and discussed.

6.1. Example of a non-linear performance function with two random variables

This example involves a non-linear analytical equation of the performance function as follows:

$$G = 0.4 \times (u_1 - u_2)^2 - 0.4 \times (u_2 - 5)^3 - 10 \tag{20}$$

where u_1 and u_2 are two standard normal random variables. Firstly, only the enrichment process (i.e. the training points selection) of the proposed method was investigated. The same stopping condition used in AK-MCS method (i.e. the U-criterion) was adopted. The effect of the stopping condition will be presented in a subsequent section. Finally, the evolution of the LSS during the different iterations of the enrichment process is presented and discussed.

6.1.1. Effect of the training points selection

After the generation of a Monte-Carlo population of 500, 000 points (where each point consists of two standard Gaussian random variables herein), an initial DoE of seven points was randomly selected from the generated points (cf. Fig. 4). This small DoE was found sufficient to construct a preliminary Kriging metamodel for the present performance function as may be shown from Fig. 4.

Fig. 1 presents a comparison between the enrichment strategy of the AK-MCS method and that of the AK-MCSd approach. One can observe

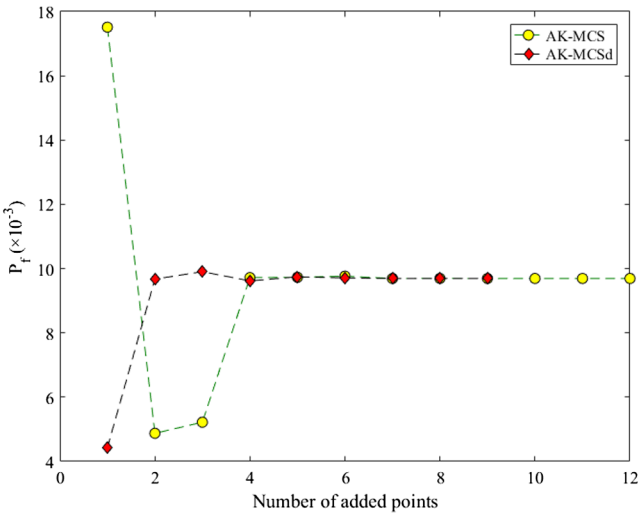

Fig. 1. Comparison between the enrichment strategies of AK-MCS and AK-MCSd.

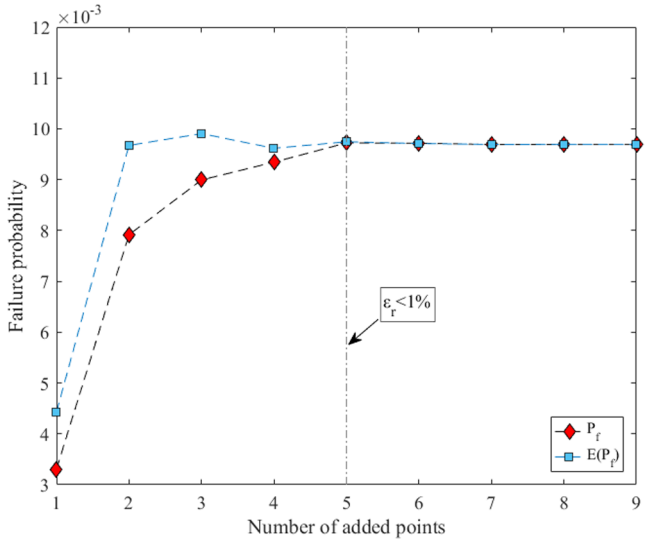

Fig. 2. Evolution of P_f and $E(P_f)$ with the number of added points.

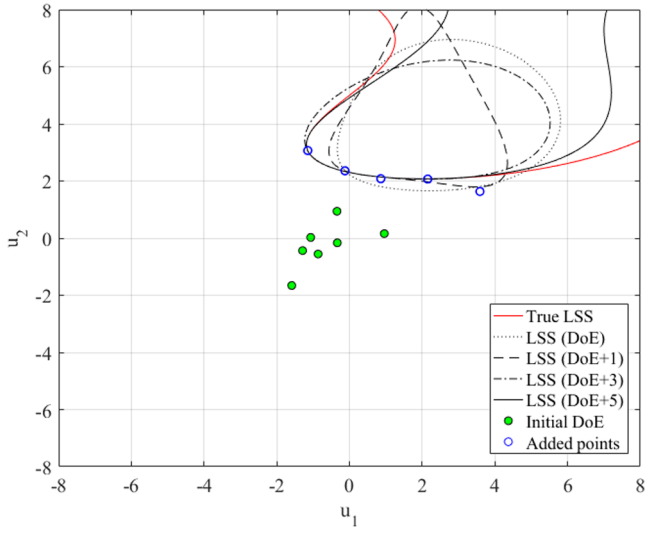

Fig. 4. Evolution of the limit state surface with the number of added points.

that the enrichment strategy of AK-MCSd method is more efficient than that of AK-MCS since it leads to a quasi-similar value of the failure probability with a reduced number of added points (9 added points in AK-MCSd instead of 12 added points in AK-MCS). This may be explained by the fact that AK-MCSd enrichment strategy, which considers the dependency between the candidate points, leads to better selected points for training.

6.1.2. *Effect of the stopping condition*

The effect of the proposed stopping criterion was investigated in this section.

Fig. 2 presents the evolution of the failure probability estimate P_f [as given by Eq. (10)] and its expectation $E(P_f)$ [as given by Eq. (13)] with the number of added points, and Fig. 3 presents the evolution of the corresponding error ε_r as defined in Eq. (18).

One can notice that (i) the values of P_f and $E(P_f)$ become closer to each other (cf. Fig. 2) and (ii) the corresponding error ε_r decreases (cf. Fig. 3), as the number of added points increases. From Figs. 2 and 3, one may observe that when using the proposed stopping criterion (and by adopting a threshold value on the error of 1%), only 5 added points were required during the enrichment process to attain the convergence of the failure probability. Notice however that by adopting the U-criterion of AK-MCS approach, 9 added points were required for which an extremely low value of the error ($\varepsilon_r = 0.0007\%$) was attained.

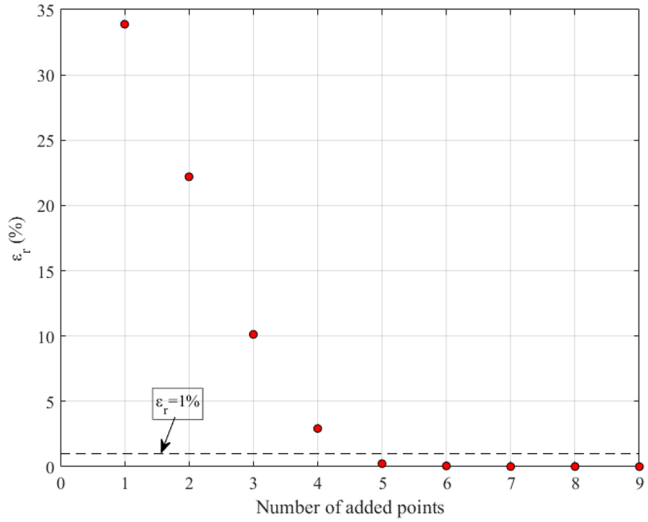

Fig. 3. Evolution of the error on the failure probability with the number of added points.

As a conclusion, the adopted AK-MCSd approach was found to be very efficient with respect to AK-MCS approach since it leads to quasi similar values of P_f ($P_f = 9.72 \times 10^{-3}$ in AK-MCSd and $P_f = 9.69 \times 10^{-3}$ in AK-MCS) using a much reduced number of added points (5 added points in AK-MCSd instead of 12 added points in AK-MCS). It should be noted that a very close value of $P_f = 9.802 \times 10^{-3}$ (with a corresponding coefficient of variation of 1.42%) was obtained when using the crude MCS methodology based on 500, 000 simulations.

6.1.3. *Evolution of the limit state surface with the number of added points*

Fig. 4 presents the evolution of the LSS with the number of added points. In this figure, the points corresponding to the initial DoE are presented in green filled circles; the five added points being presented in blue circles. The LSS of the preliminary constructed metamodel based on the initial DoE is presented in dotted black line. The LSS is also presented at three different iterations of the enrichment process (initial DoE + 1 added point, initial DoE + 3 added points and initial DoE + 5 added points). Notice finally that the true LSS (i.e. that corresponding to the true performance function G) is presented in continuous red line on the same figure. As may be seen from this figure, the LSS of the preliminary Kriging metamodel is very different from the LSS of the true performance function. This LSS progressively improves with the number of added points. At the end of the enrichment process, the LSS corresponding to the obtained metamodel (in continuous black) is very well matched with that of the true performance function (in continuous red), especially in the zone where the probability density is not negligible.

6.2. *Example of a series system involving four limit state functions*

This example involves a series system having four limit state functions. It is given by the following equation:

$$G = min \begin{Bmatrix} 3 + 0.1 \times (u_1 - u_2)^2 \pm \frac{(u_1 + u_2)}{\sqrt{2}} \\ (u_1 - u_2) + \frac{7}{\sqrt{2}} \\ (u_2 - u_1) + \frac{7}{\sqrt{2}} \end{Bmatrix} \tag{21}$$

where u_1 and u_2 are two standard normal random variables. After the generation of a large population of 500, 000 points (where each point consists of two standard Gaussian random variables), an initial DoE of

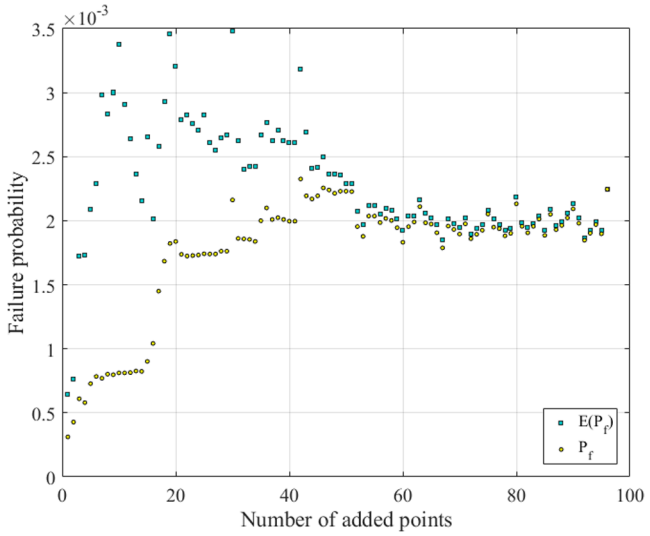

Fig. 5. Evolution of P_f and $E(P_f)$ with the number of added points.

20 points randomly selected from the generated population was found necessary to construct a preliminary Kriging metamodel for the present complex multi-branch LSS.

Fig. 5 presents the evolution of the failure probability estimate P_f and its expectation $E(P_f)$ with the number of added points, and Fig. 6 presents the evolution of the corresponding error ε_r as given by Eq. (18). One can notice that (i) the values of P_f and $E(P_f)$ become closer to each other (cf. Fig. 5) and (ii) the corresponding error ε_r decreases (cf. Fig. 6), as the number of added points increases. The error attains a value of 0.18%($< 1\%$) for a number of added points equal to 96.

6.2.1. Comparison with AK-MCS results

Fig. 7 presents the evolution of the failure probability estimate with the number of added points as given by AK-MCS and AK-MCSd approaches. One can observe that AK-MCSd method is more efficient than AK-MCS since it leads to a quasi-similar value of the failure probability ($P_f = 2.24 \times 10^{-3}$ using AK-MCSd and $P_f = 2.23 \times 10^{-3}$ using AK-MCS) with a reduced number of added points (96 added points in AK-MCSd instead of 182 added points in AK-MCS). It should be noted that a very close value of $P_f = 2.22 \times 10^{-3}$ (with a corresponding coefficient of variation of 2.99%) was obtained when using the crude MCS based on 500, 000 simulations.

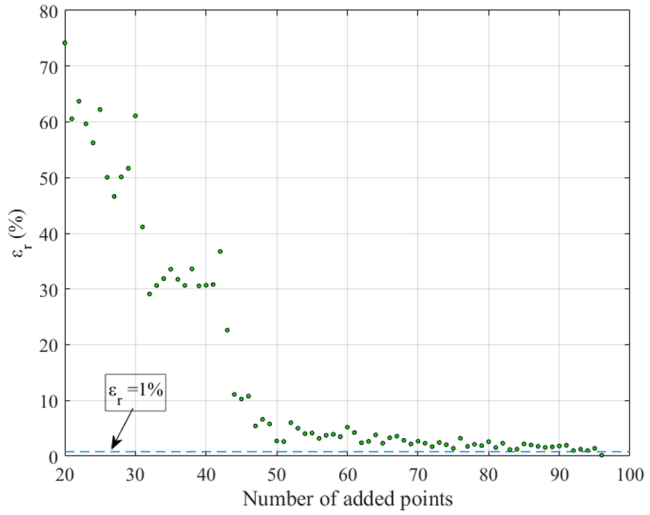

Fig. 6. Evolution of the error on the failure probability with the number of added points.

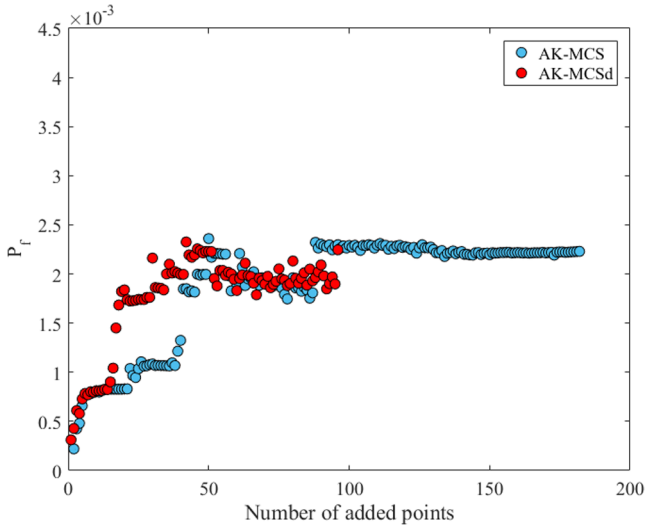

Fig. 7. Comparison between AK-MCS and AK-MCSd approaches.

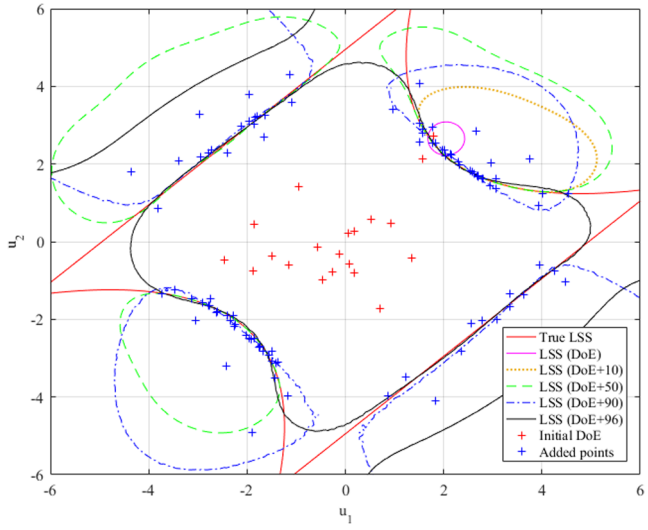

Fig. 8. Evolution of the limit state surface with the number of added points.

6.2.2. Evolution of the limit state surface with the number of added points

Fig. 8 aims at presenting the evolution of the LSS with the number of added points. The points corresponding to the initial DoE and those corresponding to the added points are presented in this figure in red crosses and in blue crosses respectively. The LSS of the preliminary constructed metamodel (based on the initial DoE) and the true LSS (i.e. that corresponding to the true performance function) are plotted on the same figure. Fig. 8 also shows the LSSs of the metamodel corresponding to four different iterations of the enrichment process [i.e. initial DoE+ 10 added points, initial DoE+ 50 added points, initial DoE+ 90 added points and initial DoE+ 96 added points].

As may be seen from Fig. 8, the LSS of the preliminary Kriging metamodel is extremely different from that of the true performance function. Notice however that this LSS progressively improves with the number of added points. At the end of the enrichment process (i.e. for 96 added points), the LSS corresponding to the obtained metamodel (in continuous black) is well matched with that of the true performance function (in continuous red), especially in the zones where the probability density is not negligible (i.e. in the zones that are the most close to the origin of the standard coordinate system). The proposed method is thus shown to be very efficient even when dealing with complex limit state surfaces.

It should be noted that AK-MCSd approach is an efficient approach

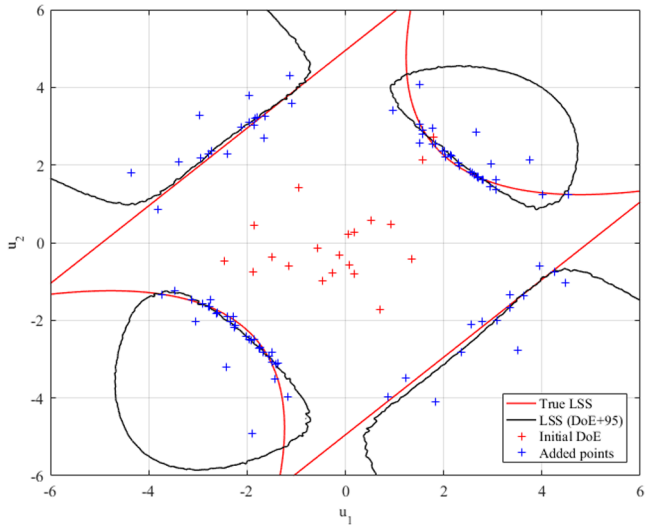

Fig. 9. Comparison between the LSS obtained one enrichment step before satisfying the proposed stopping criterion (i.e. for 95 added points) and that of the true performance function.

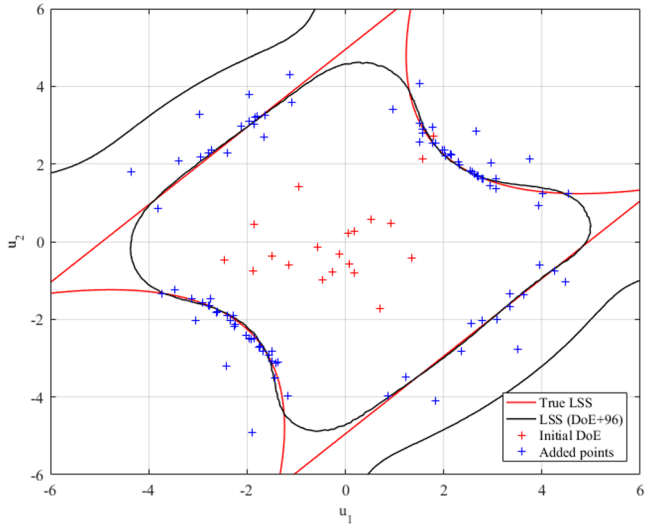

Fig. 10. Comparison between the LSS of the obtained metamodel (i.e. for 96 added points) and that of the true performance function.

since it can detect the convergence of the failure probability from its early beginning. Indeed, Fig. 9 presents a comparison between the LSS obtained one enrichment step before the proposed stopping criterion is satisfied (i.e. for 95 added points) and that of the true performance function plotted in red on the same figure. One can notice that at this stage, the LSS is clearly not well matched with the true LSS. Fig. 10 presents a comparison between the LSS of the obtained metamodel after satistfying the proposed stopping criterion (i.e. for 96 added points) with that of the true performance function.

One can observe that the LSS of the metamodel is well but not perfectly matched with that of the true performance function. Indeed, by continuing the enrichment process until attaining the U-criterion (i.e. for 155 added points in this example), the metamodel continues to be improved as is shown in Fig. 11. However, the obtained LSS by AK-MCSd approach (i.e. for 96 added points) is considered to be sufficient to accurately determine the failure probability; the newly added points (i.e. $155 - 96 = 59$ points) being unnecessary training points.

Table 1 presents the value of the failure probability estimate P_f and the corresponding value of $COV(P_f)$ together with value of the error ε_r as obtained from AK-MCSd using different stopping criteria (cf. the first

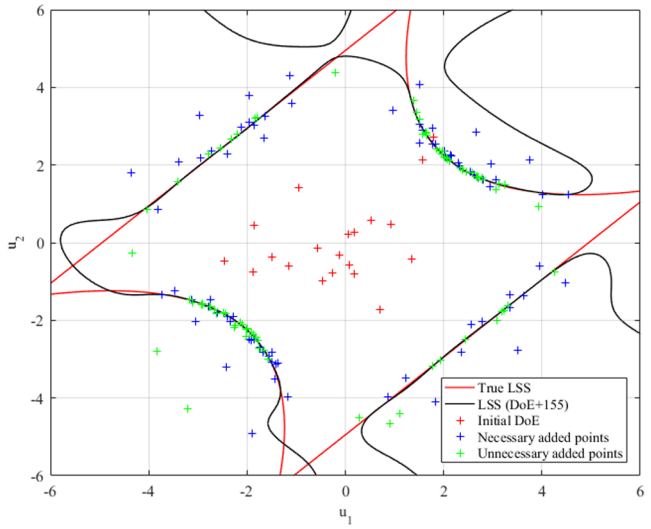

Fig. 11. Comparison between the LSS obtained by adopting the U-criterion (i.e. for 155 added points) and that of the true performance function.

three lines of this table). This table also provides the values of P_f and $COV(P_f)$ as obtained using AK-MCS approach and the crude MCS (cf. the last two lines of this table). The number of added points together with the error on P_f with respect to the value provided by the crude MCS methodology are also given in this table (cf. the last two columns of this table).

From Table 1, one can see that for 95 added points (corresponding to one enrichment step before satisfying the ε_r-stopping criterion), the value of the failure probability is not yet well predicted by the Kriging metamodel (with an error of 14.66% with respect to P_f value obtained by the crude MCS). However, after satisfying the proposed stopping criterion (i.e. for 96 added points), the value of the failure probability becomes well predicted by the metamodel (with an error of 0.99% with respect to P_f value obtained by the crude MCS). The reason why the value of P_f corresponding to 96 added points (i.e. when the stopping criterion is satisfied) is a bit far from that corresponding to 95 added points (i.e. one step before) [cf. Fig. 5 and table 1] may be explained by the evolution of the LSS of the constructed metamodel. This LSS has undergone a non-negligible improvement in the zone of interest for the computation of the failure probability when the number of added points increases from 95 to 96 as may be shown from Figs. 9 and 10. Indeed, this improvement in the form of the LSS has influenced the classification of several MC points (safe/failure) and has led to a relatively significant change in the value of P_f. This phenomenon was observed herein because of the complex multi-branch limit state surface considered in this example and it may explain the reason why the convergence of P_f to the "true" value is not very smooth until the end of the enrichment process.

From Table 1, it can also be seen that a further improvement in the metamodel beyond 96 added points (by adding new training points until reaching the U-criterion, i.e. for 155 added points) leads to a quasi-similar value of the failure probability as the case of 96 added points, thus proving the convergence of P_f beyond this number of added points. The additional improvement of the metamodel is thus shown to be unnecessary as the cost of the extra evaluations induced by the newly added training points is with no benefit in terms of the improvement in the value of the failure probability. The preceeding observations confirm the accuracy of AK-MCSd approach with the corresponding stopping condition adopted in this paper.

6.3. *Example of a roof truss structure*

This example application consists of a roof truss structure problem

Table 1
$\mathbf{P_f}$, $\mathbf{COV}(\mathbf{P_f})$ and $\boldsymbol{\varepsilon_r}$ as obtained from AK-MCSd methodology using different stopping criteria, together with the values of $\mathbf{P_f}$ and $\mathbf{COV}(\mathbf{P_f})$ as obtained using AK-MCS and the crude MCS approaches.

Method	$P_f \times 10^{-3}$	$COV\,(P_f)$ (%)	ε_r(%)(Eq. (18))	Number of added points	Error with respect to MCS (%)
AK-MCSd (one step before satisfying ε_r criterion)	1.898	3.243	1.41	95	14.66
AK-MCSd + ε_r criterion	2.246	2.980	0.18	96	0.99
AK-MCSd + U-criterion	2.220	2.998	2.52×10^{-4}	155	0.18
AK-MCS	2.230	2.991	–	182	0.27
Crude MCS	2.224	2.995	–	–	–

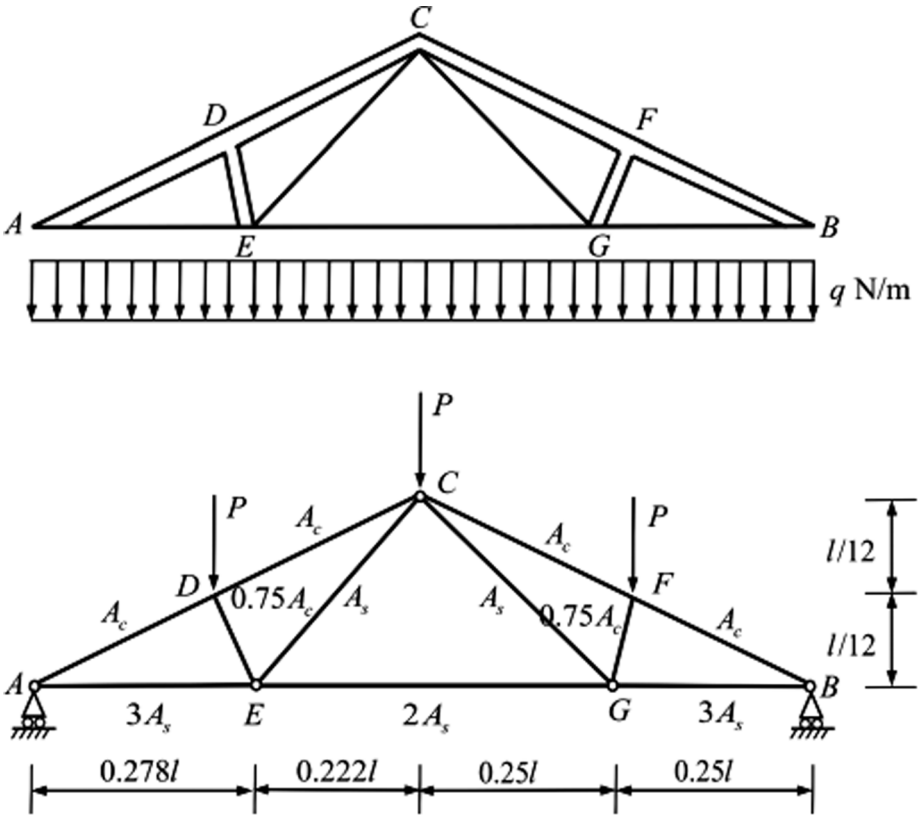

Fig. 12. Roof truss structure [28].

(cf. Fig. 12) where the performance function involves six random variables. This problem was solved by Zhu and Du [28] in order to check the efficiency of their approach. The main purpose herein is to show the improvement provided by the proposed AK-MCSd approach with respect to the approach proposed by these authors [28].

Assume that the truss bars bear a uniformly distributed load q that can be transformed into nodal load $P = ql/4$. The perpendicular deflection of the truss peak node C is computed by the following equation:

$$\Delta C = \frac{ql^2}{2}\left(\frac{3.81}{A_c E_c} + \frac{1.13}{A_s E_s}\right) \tag{22}$$

where A_c and A_s are respectively the cross-sectional areas of the reinforced concrete and steel bars, E_c and E_s are their corresponding elastic modulus, and l is the length of the truss. A failure event occurs when the perpendicular deflection ΔC exceeds 3 cm. The performance function is given as follows:

$$G = 0.03 - \frac{ql^2}{2}\left(\frac{3.81}{A_c E_c} + \frac{1.13}{A_s E_s}\right) \tag{23}$$

where (q, l, A_s, A_c, E_s, E_c) are normal independent variables. The corresponding distribution parameters are given in Table 2.

Twenty runs of the proposed AK-MCSd approach were performed for this example in order to be compatible with the work by Zhu and Du [28]. For each run, an initial DoE of 12 points was adopted. Figs. 13 and 14 present respectively the evolution of the failure probability estimate and the error on this failure probability with the number of added points for the 20 runs. One can observe from Fig. 14 the decreasing trend of the error for each run until reaching a value smaller than 1%, thus indicating the convergence of the failure probability estimate.

Table 3 shows the average results from 20 runs. From this table, one can observe that the proposed AK-MCSd approach provides an accurate value of the failure probability (averaged over 20 runs) making use of a much reduced number of calls to the true performance function as

Table 2
Random variables of the roof truss structure example.

Variable	Mean	Standard deviation
q	20,000	1400
l	12	0.12
A_s	9.82×10^{-4}	5.982×10^{-5}
A_c	0.04	0.0048
E_s	1×10^{11}	6×10^{9}
E_c	2×10^{10}	1.2×10^{9}

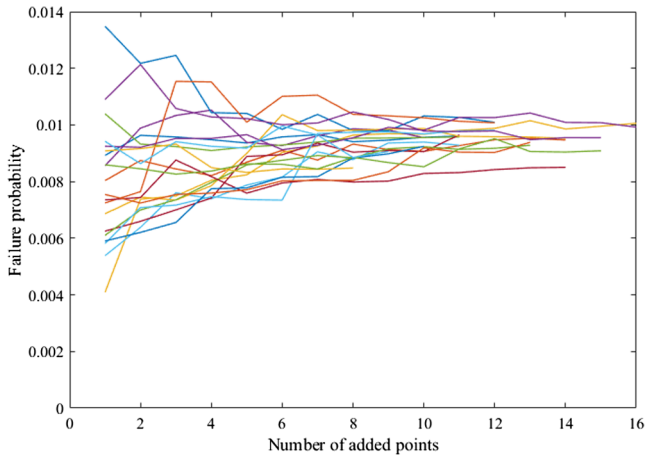

Fig. 13. Evolution of the failure probability with the number of added points for 20 runs.

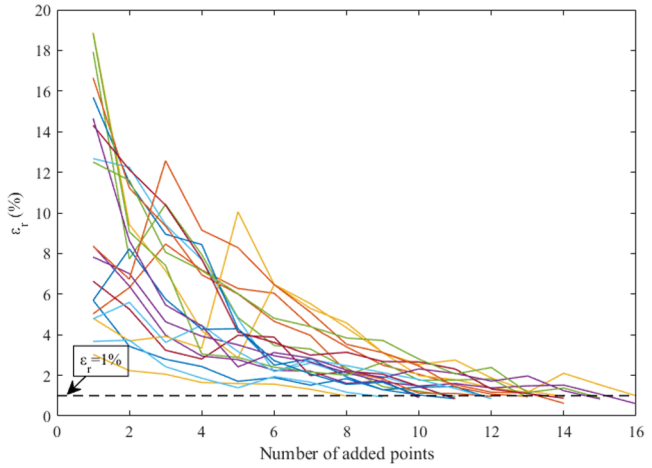

Fig. 14. Evolution of the error on the failure probability with the number of added points for 20 runs.

Table 3
Average results from 20 runs.

Method	P_f ($\times 10^{-3}$)	Number of calls	Error with respect to MCS (%)
MCS [28]	9.4890	2×10^6	–
AK-MCS [28]	9.3935	92.40	1.01
Zhu and Du [28]	9.5482	43.25	0.62
AK-MCSd	9.4610	24.35	0.29

compared to the approach proposed by [28]. The reduction in the number of calls within the present approach is equal to 44%. Thus, AK-MCSd approach may be considered more efficient than the approach proposed by [28].

6.4. Appilaction to a problem of a monopile foundation embedded in a spatially varying soil

AK-MCSd approach was applied in this section to a geotechnical problem involving a monopile foundation embedded in a spatially varying clayey soil and subjected to a combined loading.

The numerical model is presented in Fig. 15. The monopile consists of a 3 m diameter open-ended steel pile having a thickness of 5 cm and an embedment depth L of 18 m. The soil consists of an undrained normally consolidated clay. The soil undrained cohesion was considered as a 1-D vertical random field defined by a constant mean of

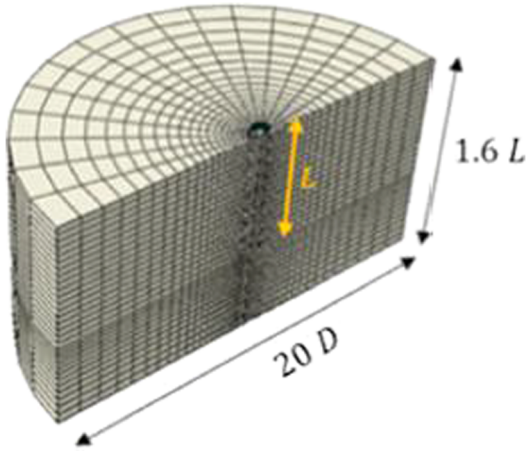

Fig. 15. Monopile foundation model [after El Haj et al. (2019)].

50 kPa, a coefficient of variation of 10% and a square exponential autocorrelation function with a corresponding autocorrelation distance of 2 m. The generation of the cohesion random field was performed by EOLE discretization method making use of 16 standard Gaussian random variables. Details on the numerical modeling of the mechanical problem and the discretization of the cohesion random field are not provided herein to avoid repetition. Interested readers may refer to [48].

Notice that the present problem can be considered as a black-box costly-to-evaluate finite element model. It requires 15 min per run. The corresponding performance function depends on 16 standard Gaussian random variables. The use of a Kriging-based approach for the probabilistic analysis is of great interest herein because it allows one to substitute the unknown performance function of the time-demanding black-box numerical code by a metamodel on the basis of a few number of runs of the finite element software, which may enormously reduce the computational cost of the probabilistic analysis. Notice also that a Monte Carlo simulation is not straightforward for such a problem and it requires a huge computational effort. Indeed, a crude MCS with a population of 500,000 points (or realizations of the random field) would take about 125,000 h (≈5208 days) to be performed.

6.4.1. Comparison between AK-MCSd and AK-MCS results

The aim of this sub-section is the comparison between the performance of the proposed AK-MCSd approach and that of the classical AK-MCS approach as applied to the present practical geotechnical problem in order to quantify the benefit that can be afforded by AK-MCSd in terms of computational effort.

An initial DoE of 15 points (close to the problem stochastic dimension of 16 random variables) was randomly selected from a large MC population of 500,000 points. Indeed, for geotechnical engineering problems involving spatially varying soil properties (with a relatively high stochastic dimension i.e., with a relatively high number of random variables), it was found that an initial DoE size that is close to the problem stochastic dimension may be a suitable choice to capture the LSS of the studied problem (cf. Soubra et al. [49]).

Figs. 16 and 17 present respectively the evolution of the failure probability and the corresponding coefficient of variation with the number of added points as obtained by AK-MCS and AK-MCSd approaches. One can observe that AK-MCSd method is more efficient than AK-MCS since it leads to a quasi-similar value of the failure probability with a reduced number of added points. As may be seen from Table 4, a failure probability $P_f = 3.40 \times 10^{-3}$ is obtained when using AK-MCSd requiring 186 added points and a corresponding computational time of about 44.54 h while a very close value of $P_f = 3.41 \times 10^{-3}$ is obtained when using AK-MCS with a much larger number of added points equal to 440 and a corresponding computational time of about 178.18 h. The

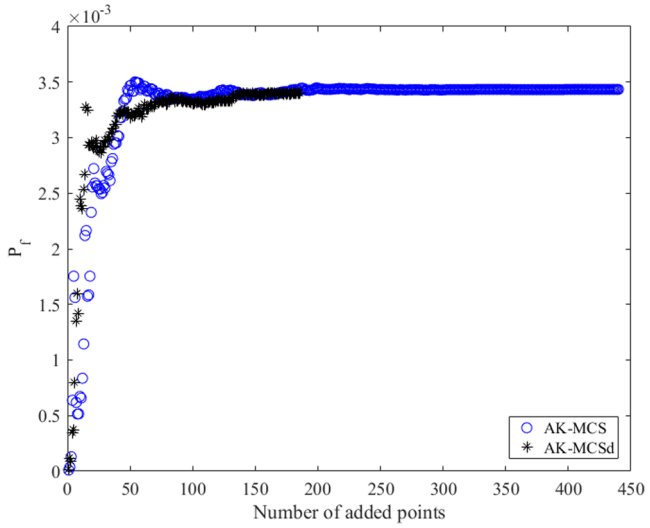

Fig. 16. Failure probability versus the number of added points.

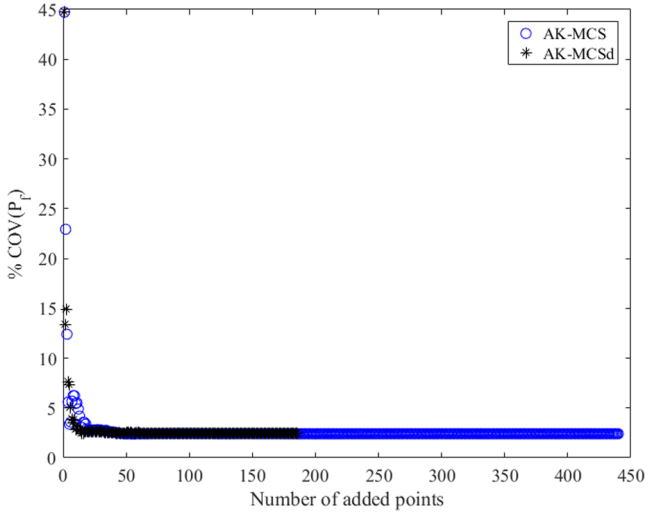

Fig. 17. Coefficient of variation of the failure probability versus the number of added points.

Table 4
Comparison between AK-MCS and AK-MCSd results.

Method	$P_f \times 10^{-3}$	% COV (P_f)	Nb. of added points	Time (h)
AK-MCS	3.41	2.42	440	178.18
AK-MCSd	3.40	2.42	186	44.54
Crude MCS	-	-	-	≈125,000

coefficient of variation of the failure probability as obtained by both approaches is very small (about 2.42%) thus indicating the accuracy of the obtained results.

Fig. 18 presents the evolution of the failure probability estimate P_f and its expectation $E(P_f)$ with the number of added points, and Fig. 19 presents the evolution of the corresponding error ε_r as given by Eq. (18). One can notice that (i) the values of P_f and $E(P_f)$ become closer to each other (cf. Fig. 18) and (ii) the corresponding error ε_r decreases (cf. Fig. 19), as the number of added points increases. This error attains a value of 0.98%($< 1\%$) for a number of added points equal to 186.

6.4.2. Performance of FORM approximation method for problems involving spatially varying soil properties

The aim of this sub-section is to investigate the effect of the vertical

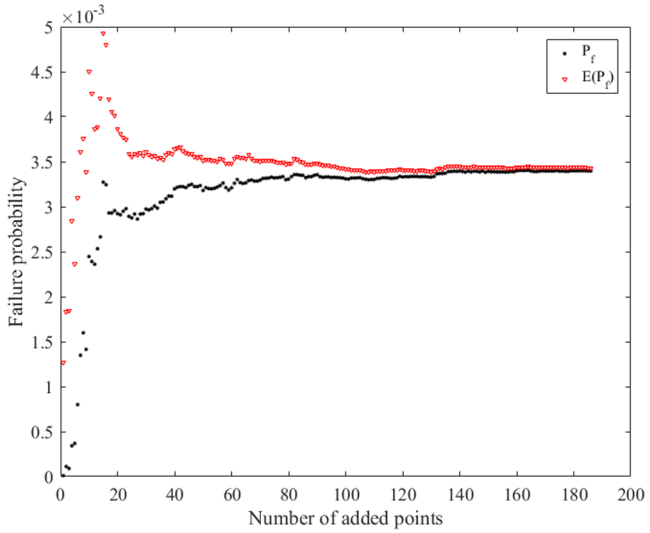

Fig. 18. Evolution of P_f and $E(P_f)$ with the number of added points.

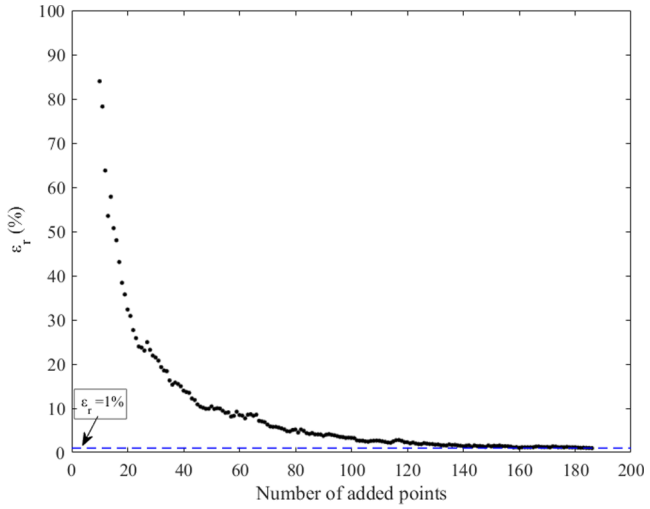

Fig. 19. Error on the failure probability versus the number of added points.

autocorrelation distance (or the degree of soil heterogeneity in the vertical direction) on the performance of FORM approximation method. This task is of great importance since it allows one to identify the cases where the proposed AK-MCSd probabilistic approach should be used. Indeed, the estimation of the failure probability for the cases corresponding to a low stochastic dimension and a nearly linear LSS may be simply performed using approximation methods such as FORM or SORM [2] without the need to resort to more sophisticated approaches.

Fig. 20 presents the relative error between the failure probability obtained using AK-MCSd and that obtained from the following formula using FORM approximation:

$$P_f^{\text{FORM}} \approx \Phi(-\beta_{HL}) \tag{24}$$

where β_{HL} in this equation is the Hasofer-Lind reliability index. Notice that the Hasofer-Lind reliability index can be easily obtained from the Kriging metamodel of the performance function since the metamodel is expressed in the standard Gaussian space of random variables.

Fig. 20 shows that the error between the failure probability obtained using the proposed AK-MCSd approach and that obtained using FORM approximation is high for the cases of small autocorrelation distances corresponding to high soil heterogeneity. This error decreases with the increase in the autocorrelation distance and tends to a small constant value (smaller than 2%) once the autocorrelation distance

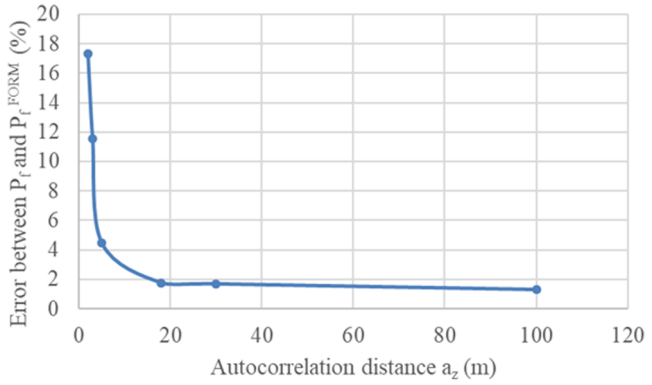

Fig. 20. Error with respect to FORM approximate method.

Table 5
Effect of increasing the number of candidate points within AK-MCSd.

Number of candidate points	P_f ($\times 10^{-3}$)	Number of added points	Time (h)
20	3.40	186	44.54
100	3.42	132	33.98
200	3.47	120	30.03
300	3.47	120	30.21

becomes close or higher than the monopile embedded length (i.e. 18 m). Thus, FORM approximation method may lead to inaccurate results for the cases involving small values of the autocorrelation distance.

As a conclusion, the proposed AK-MCSd approach is of great interest for the cases where the vertical autocorrelation distance is smaller than the monopile embedded depth; the approximate methods being not accurate for the problems involving a relatively high spatial variability.

6.4.3. Effect of the number of candidate points

This section aims at studying the effect of increasing the number of candidate points within AK-MCSd approach on the number of added points. Indeed, this number may improve the selection of the enrichment points and lead to a further reduction in the number of added points. Three tests were performed for the present geotechnical problem where an increased number of candidate points is used for each test (100, 200 and 300 candidate points instead of 20 points). The results are shown in Table 5.

From this Table, one may see that when increasing the number of candidate points from 20 points to 200 points, the number of added points decreases from 186 to 120 points and the computational time decreases from 44.54 h to 30.03 h. Notice however that a further increase in the number of candidate points (i.e. for 300 candidate points) was shown to be detrimental to the computation time. This may be explained by the fact that when the number of candidate points exceeds a certain limit n_{lim}, the number of added points stops to decrease (i.e. it remains equal to 120 points) and the time required by the algorithm to find the best point naturally increases with the increase in the number of candiate points. Thus, the number of 200 candidate points seems to be the optimal one for the present problem in terms of computational cost.

7. Conclusions

An improved Kriging-based probabilistic approach was proposed in this paper for practical use in reliability engineering. The proposed approach is a variant of AK-MCS method developed by [1] where some improvements were introduced to the learning process in the aim of increasing the efficiency of the failure probability estimation. An enhanced methodology of selection of a training point that considers the complete output of the Kriging process was suggested. Furthermore, a new stopping condition was proposed. This stopping condition is based on the gap between the estimated value of the failure probability and the expectation of this failure probability that takes into account the uncertainty of the Kriging predictions.

The proposed approach was shown to lead to quasi-similar probabilistic results as compared to the classical AK-MCS approach when dealing with a nonlinear or a complex limit state function, with a much reduced number of evaluations of the true performance function. It was also demonstrated that the proposed approach is of great interest for geotechnical engineering problems involving spatially varying soil properties with small values of the autocorrelation distance, the approximate methods being inaccurate for these configurations.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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