Computers and Geotechnics 114 (2019) 103107

Contents lists available at ScienceDirect

Computers and Geotechnics

journal homepage: www.elsevier.com/locate/compgeo

Research Paper

Probabilistic analysis of strip footings resting on spatially varying soils using kriging metamodeling and importance sampling

Abdul-Hamid Soubra[a,*], Tamara Al-Bittar[b], Jawad Thajeel[c], Ashraf Ahmed[d]

[a] Civil Engineering Department, University of Nantes, Boulevard de l'université, CS 70152, 44603 Saint-Nazaire Cedex, France
[b] Scientific Research Center in Engineering, Faculty of Engineering, Branch I, Lebanese University, Haykalieh, Al Koura, Lebanon
[c] Civil Engineering Department, College of Engineering, University of Thi-Qar, Iraq
[d] Civil Engineering Department, Faculty of Engineering, Aswan University, Aswan, Egypt

ARTICLE INFO

Keywords:
Strip footing
Bearing capacity
Spatial variability
Kriging
Importance sampling
Probability of failure

ABSTRACT

This paper aims at computing the probability of failure of strip footings resting on a spatially varying soil and subjected to a vertical load. The active learning reliability method (called AK-IS) which is a combination of kriging metamodeling and importance sampling (IS) is used. The AK-IS technique significantly reduces the computation time with respect to the classical active learning reliability technique (called AK-MCS) combining kriging with Monte Carlo Simulations (MCS) by sampling around the design point. It was shown that the critical realization corresponding to the design point exhibits a perfect symmetry about the central vertical axis of the foundation.

1. Introduction

The computation of the failure probability P_f of geotechnical structures is generally performed in literature using the crude Monte Carlo Simulations (MCS) or a variance reduction technique (e.g [1–11]). Despite of being robust and accurate, MCS shows a low efficiency when considering practical problems with small P_f values especially if a small value of the coefficient of variation on this failure probability is desired. This is due to the large population required in this case thus leading to a significant number of evaluations of the performance function. For instance, one million of samples are required for the computation of P_f values in the order of 10^{-4} for a coefficient of variation on P_f of 10%. Furthermore, when dealing with spatially varying soil properties as is the case in the present paper, the evaluation of the performance function is generally based on computationally expensive finite element/finite difference codes. This naturally leads to a high computational cost. The variance reduction techniques such as subset simulation (SS) or asymptotic sampling (AS) significantly reduce the required number of evaluations of the performance function with respect to the crude MCS; however, these methods remain quite expensive for the computation of the failure probability of computationally-expensive mechanical models.

Recently, several metamodeling techniques have been developed for the probabilistic analysis of engineering systems such as the polynomial chaos expansion and its extension the sparse polynomial chaos expansion, the artificial neural networks, the support vector machine and the kriging. These techniques have shown high efficiency when the user is interested in the computation of the first two statistical moments (i.e. the mean and the standard deviation) of the system response (e.g. [12–15]). Notice however that for problems involving the computation of small failure probabilities, a large set of sample points is required to accurately construct the meta-model in the zone of interest for the computation of the failure probability (i.e. the tail distribution). This task is time-consuming when the performance function is evaluated using a computationally expensive finite element/finite difference code as is the case in the present paper.

In order to overcome the shortcoming of the above-mentioned methods related to the large number of calls to the mechanical model, a combined use of a metamodeling technique with a simulation-based method (e.g. Monte Carlo, importance sampling, subset simulation) was proposed by several authors (cf. [16–22]). Among these methods, a combined use of a kriging metamodeling technique with a Monte Carlo Simulation (MCS) methodology was suggested by Echard et al. [18]. This method is an active learning reliability method combining kriging and Monte Carlo simulation (called AK-MCS). It overcomes the shortcomings of the crude MCS and the kriging metamodeling technique when used separately. This method consists in constructing a meta-model (i.e. an analytical equation which substitutes the original mechanical model) based on a relatively small number of calls to the computationally expensive mechanical model. The computation of the

* Corresponding author.
E-mail addresses: Abed.Soubra@univ-nantes.fr (A.-H. Soubra), tamara.albittar1@ul.edu.lb (T. Al-Bittar), Ashraf.ahmed@aswu.edu.eg (A. Ahmed).

https://doi.org/10.1016/j.compgeo.2019.103107
Received 30 August 2018; Received in revised form 20 May 2019; Accepted 22 May 2019
Available online 05 July 2019
0266-352X/ © 2019 Elsevier Ltd. All rights reserved.

failure probability may thus be easily performed using this meta-model. It should be emphasized here that AK-MCS makes use of a powerful learning function (based on the kriging mean prediction and the kriging variance prediction) for the selection of the 'best' samples to be evaluated by the computationally expensive mechanical model.

Echard et al. [18] have illustrated the efficiency of the AK-MCS method through the computation of the failure probability for some academic examples for which the system response is known analytically (i.e. where the computation time of the corresponding performance function is quasi-negligible). Later on, [23] used the AK-MCS technique by Echard et al. [18] for the computation of the failure probability against soil punching of a strip footing resting on a spatially varying soil. This problem required the use of a computationally expensive mechanical model based on numerical simulations for the computation of the performance function. A much reduced number of calls to the mechanical model was obtained when using the AK-MCS method as compared to the commonly used variance reduction techniques.

Although AK-MCS significantly reduces the computation time with respect to the variance reduction techniques and the meta-modeling techniques, the computation time of this method remains important. This is because the kriging predictions (mean prediction and variance prediction) *via* the meta-model should be evaluated for the whole Monte Carlo population each time a 'best' new sample (called hereafter added sample) is to be selected for evaluation by the mechanical model. This makes the AK-MCS time-consuming especially when dealing with the small practical values of the failure probability. This statement was also reported by Echard et al. [18]. In order to overcome the inconvenience related to the huge number of predictions by the meta-model, this paper makes use of the active learning method combining kriging with importance sampling IS (called AK-IS procedure) suggested by Echard et al. [19]. The aim is to perform a probabilistic analysis of the same problem considered by [23] with a more powerful probabilistic technique. In the framework of this approach, the small failure probability can be estimated with a similar accuracy as AK-MCS but using a much smaller size of the population (i.e. a much smaller number of calls to the kriging meta-model each time a new sample is to be selected for evaluation by the mechanical model) because the sampling population is centered at the design point. This reduced number of calls to the kriging meta-model naturally leads to a reduction in the computation time with respect to AK-MCS approach especially for the very small values of the failure probability that require a significant number of added samples.

Contrarily to [19] where the determination of the design point for importance sampling computation is straightforward (because the performance function used by these authors was given by an analytical equation), the computation of the design point becomes an issue in the present case of spatially varying soil properties where an analytically-unknown performance function with several random variables is involved in the analysis. This paper presents a simple and non-expensive iterative procedure based on kriging metamodeling for the determination of the design point in the present case of a spatially varying soil medium characterized by a quite large number of random variables. This is followed by the enrichment process to lead to a sufficiently accurate meta-model for the computation of the failure probability.

The soil cohesion and angle of internal friction were considered as random fields. The Expansion Optimal Linear Estimation (EOLE) methodology was used to generate these two random fields. As mentioned above, the mechanical model used in the probabilistic analysis was the one presented in [23]. It is based on numerical simulations using the finite difference code FLAC^{3D}. The same deterministic and uncertain parameters considered in [23] were also conserved in this paper for comparison purposes.

The paper is organized as follows: The next two sections aim at presenting EOLE methodology and the proposed AK-IS procedure in the case of geotechnical structures involving spatially varying soil properties. This is followed by the probabilistic results. After a validation of the present AK-IS approach *via* a simple academic example, some probabilistic results involving a strip footing resting on a spatially varying soil are presented and discussed. The paper ends with a conclusion of the main findings.

2. The Expansion Optimal Linear Estimation (EOLE) methodology

The Expansion Optimal Linear Estimation (EOLE) method by [24] is used herein to discretize the two random fields of c and φ. The present two random fields are denoted by $Z_i^{NG}(x, y)$ $(i = c, \varphi)$. They are described by two non-Gaussian marginal cumulative density functions G_i $(i = c, \varphi)$ and a common square exponential autocorrelation function ρ_Z^{NG} [(x, y), (x', y')] as follows:

$$\rho_Z^{NG}[(x, y), (x', y')] = \exp\left(-\left(\frac{|x - x'|}{a_x}\right)^2 - \left(\frac{|y - y'|}{a_y}\right)^2\right) \tag{1}$$

where a_x and a_y are the autocorrelation distances along x and y respectively.

In the present discretization method, one should first define a stochastic grid composed of N_q grid points (or nodes). The common non-Gaussian autocorrelation matrix $\Sigma_{\chi;\chi}^{NG}$ computed using Eq. (1) should be transformed into the Gaussian space using the Nataf correction functions proposed by [25]. As a result, one obtains two Gaussian autocorrelation matrices $\Sigma_{\chi;\chi}^{c}$ and $\Sigma_{\chi;\chi}^{\varphi}$ that can be used to discretize the two Gaussian random fields at any point using the following equations:

$$\widetilde{Z}_i(x, y) = \mu_i + \sigma_i \sum_{j=1}^{M} \frac{\xi_j^i}{\sqrt{\lambda_j^i}}.\ (\phi_j^i)^T.\ \Sigma_{Z(x,y);\chi}^i \quad i = c, \varphi \tag{2}$$

where μ_i and σ_i $(i = c, \varphi)$ are respectively the mean and standard deviation values of the two random fields, ξ_j^i $(i = c, \varphi; j = 1, \ldots, M)$ are two blocks of independent standard normal random variables, λ_j^i, ϕ_j^i $(i = c, \varphi; j = 1, \ldots, M)$ are the eigenvalues and eigenvectors of the two Gaussian autocorrelation matrices $\Sigma_{\chi;\chi}^{c}$ and $\Sigma_{\chi;\chi}^{\varphi}$ respectively, $\Sigma_{Z(x,y);\chi}^i$ is the correlation vector between the values of the random field at the different nodes and its value at an arbitrary point (x, y) as obtained using Eq. (1), and finally M is the number of terms (expansion order) retained in the EOLE method. This number will be determined later in this section based on the variance of the error. Once the two Gaussian random fields (i.e. Eq. (2)) are obtained, they should be transformed to the non-Gaussian space by applying the following formula:

$$\widetilde{Z}_i^{NG}(x, y) = G_i^{-1}\{\Phi[\widetilde{Z}_i(x, y)]\} i = c, \varphi \tag{3}$$

where $\Phi(\cdot)$ is the standard normal cumulative density function. It should be mentioned here that the series given by Eq. (2) are truncated for a number of terms M (expansion order) smaller than the number of grid points N_q, after sorting the eigenvalues λ_j^c and λ_j^φ $(j = 1, \ldots, N_q)$ in a descending order. This number should assure that the variance of the error is smaller than a prescribed tolerance. Notice that the variance of the error for EOLE for a given number s of terms is given by [24] as follows:

$$Var\,[Z_i(x, y) - \widetilde{Z}_i(x, y)] = \sigma_Z^2\left\{1 - \sum_{j=1}^{s} \frac{1}{\lambda_j^i}((\phi_j^i)^T \Sigma_{Z(x,y);\chi}^i)^2\right\} \quad (i = c, \varphi) \tag{4}$$

where $Z_i(x, y)$ and $\widetilde{Z}_i(x, y)$ are respectively the exact and the approximate values of the random fields at a given point (x, y). In this paper, a maximal value of 5% was adopted for the variance of the error when discretizing the two random fields (see column 3 of Table 5).

3. Proposed AK-IS procedure for geotechnical structures involving spatially varying soil properties

This paper aims at extending the AK-IS approach by [19] to the case of a spatially varying soil where the computationally expensive mechanical model based on FLAC^{3D} software is used in the analysis. Details on kriging metamodeling were not provided herein and the reader may refer to [26] or to different recently published papers where kriging metamodeling is used as in [18,19] and [23]. Also, the details on AK-IS as presented by [19] is not provided herein in order to avoid repetition. Only its extension to the case of spatially varying soil properties was presented in some details in this paper. It should be mentioned here that the random response predicted by a kriging surrogate model is a Gaussian variate $\widehat{G} \sim N(\mu_{\widehat{G}}, \sigma^2_{\widehat{G}})$ where $\mu_{\widehat{G}}$ and $\sigma^2_{\widehat{G}}$ are the mean prediction and the corresponding mean square error (kriging variance) respectively. The variances of the training samples are zero, but the variances of the other samples are always different from zero.

The present AK-IS procedure consists of two main stages. First, the most probable failure point (design point) is determined using an approximate kriging meta-model based on a small number of samples. Second, the obtained approximate kriging meta-model is successively improved *via* an enrichment process (by adding each time a new sample selected from a probability density function $h_x(X)$ centered at the design point) until reaching a sufficiently accurate meta-model for the computation of the failure probability. These two stages are described in more details in the next two subsections.

3.1. Determination of the design point

When dealing with problems that are characterized by an explicit performance function, the design point may be easily determined by minimizing the Hasofer-Lind reliability index subjected to the constraint that the performance function equal to zero (see [19]). Notice however that when dealing with analytically-unknown performance functions with several random variables (as is the case in the present work where spatially varying soil properties are involved in the analysis), the determination of the design point is less straightforward. The problem is even more difficult when a high-dimensional stochastic problem is involved (cf. [27]). Indeed, the discretization of the two random fields of c and φ leads to a significant number of standard normal random variables (between 6 and 62 random variables) as it will be shown later in this paper. The large number of random variables requires a significant number of calls to the mechanical model for the determination of the design point.

In order to determine the design point in the present work using a relatively small number of calls to the mechanical model, an iterative procedure based on kriging metamodeling was proposed. This procedure may be described as follows (see also the flowchart presented in Fig. 1):

(1) Generate a large MCS population of N_{Mc} samples (say N_{Mc} = 500,000 samples) of M standard Gaussian random variables $\{(\xi_1^1, ..., \xi_M^1), (\xi_1^2, ..., \xi_M^2), \ldots, (\xi_1^{N_{Mc}}, ..., \xi_M^{N_{Mc}})\}$ where M is the number of random variables adopted in EOLE methodology for the discretization of both c and φ. It should be emphasized here that each sample of M standard Gaussian random variables provides (when substituted into Eqs. (2) and (3)) typical spatial variations of c and φ that respect the correlation structure of these fields, i.e. the so-called 'realizations' of c and φ. The difference between the different realizations lies in the position of the weak and strong soil zones although all realizations respect the correlation structure of the corresponding random fields.
(2) From the generated population, randomly select a small number of samples (say N_1 = 20 samples) of M standard Gaussian random variables. Then, use EOLE methodology to transform each sample into realizations of c and φ that provide the spatial distribution of the soil cohesion and angle of internal friction respectively. These realizations are obtained through the computation of the values of c and φ at the centroids of the different elements of the FLAC^{3D} mesh using Eqs. (2) and (3).
(3) Use the software FLAC^{3D} to calculate the performance function value corresponding to each sample (the performance function used herein is presented later in Eq. (10) of this paper). Based on DACE toolbox, construct an initial approximate kriging meta-model in the standard space using the N_1 samples and the corresponding performance function values.
(4) Find the minimum value of the Hasofer-Lind reliability index and the corresponding design point by making use of the already-obtained kriging meta-model and by employing the Generalized Pattern Search (GPS) algorithm within the global optimization toolbox available in Matlab.
(5) Generate a small number of samples (5 samples are used in this work) of M standard Gaussian random variables according to a multivariate standard Gaussian distribution. Then, translate these samples such that the obtained samples follow a shifted multivariate Gaussian distribution having a mean vector whose components are equal to the coordinates of the design point in the standard coordinate system. After the generation of the five samples, transform each sample into realizations of c and φ that provide the spatial distribution of the soil cohesion and angle of internal friction respectively. Finally, for each one of the five samples, compute the corresponding value of the performance function using FLAC^{3D}.
(6) Construct a new kriging meta-model in the standard space using all samples of standard Gaussian random variables generated so far (i.e from step (2) to step (5)).
(7) Compute an updated Hasofer-Lind reliability index and its corresponding tentative design point using the obtained kriging meta-model.
(8) Steps 5–7 are repeated several times until the absolute difference between two successive values of the Hasofer-Lind reliability index becomes smaller than a given tolerance. The required number of iterations is denoted hereafter as N_2. Consequently, the DoE (which is considered in this paper to represent the number of samples needed to obtain the final design point) is given by DoE = $N_1 + 5 \times N_2$.

It should be emphasized that the aforementioned procedure does not intend to accurately determine the performance function over the entire design space but it focuses on the computation of the design point using a relatively small number of evaluations of the computationally expensive mechanical model. Notice that this procedure was not suggested in [23] because one does not need to determine the design point when dealing with AK-MCS approach. Notice also that the number N_1 = 20 samples used in this procedure was arbitrarily chosen as an initial guess that can be increased if necessary, the objective being the construction of an initial approximate meta-model that is suitable for the determination of a first tentative design point using a limited number of calls to the mechanical model.

3.2. Enrichment process

Further improvement of the already-obtained kriging meta-model is achieved in this stage *via* an enrichment process. Referring to Fig. 2, the enrichment process can be explained by the following steps:

(1) Generate a population of N_{IS} samples (say N_{IS} = 10,000 samples) of M random variables according to the PDF $h_x(X)$ of a multivariate standard Gaussian distribution shifted to the obtained design point, M being the number of random variables needed by EOLE methodology to discretize the two random fields c and φ. Notice that the samples generated by IS are called hereafter candidate samples. Among these samples, only a few ones are computed by the mechanical model; however, all the candidate samples are calculated by the meta-model each time a new added sample is to be selected

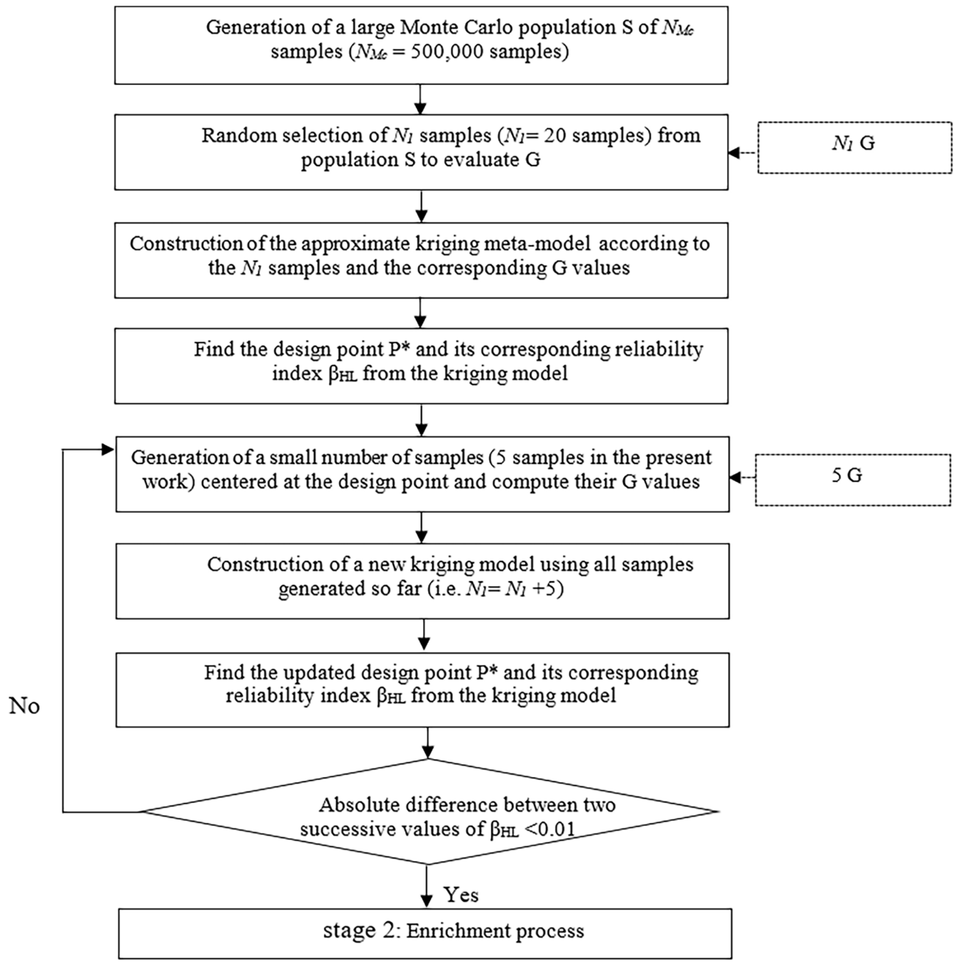

Fig. 1. Flowchart of the proposed AK-IS procedure (Stage 1: Determination of the design point).

for evaluation by the mechanical model, as it will be shown below. Notice also that the population size N_{IS} is relatively small herein as compared to the one generated in the AK-MCS procedure by [23] where N_{Mc} = 500,000 samples; however, both populations may lead to relatively close values of the coefficient of variation on P_f as it may be seen from the numerical results of AK-MCS and AK-IS approaches.

(2) Use the DACE toolbox in order to compute (for the whole population containing the N_{IS} samples) both the kriging predictor values $\mu_{\widehat{G}}$ and their corresponding kriging variance values $\sigma^2_{\widehat{G}}$ using the obtained meta-model. From the obtained values of the kriging predictors $\mu_{\widehat{G}}$ obtain an estimation of the probability of failure using the following equation:

$$P_f = \int_{\Omega} I_F(X)\frac{f_x(X)}{h_x(X)}h_x(X)dX = \frac{1}{N_{IS}}\sum_{i=1}^{N_{IS}} I_F(X_i)\frac{f_x(X_i)}{h_x(X_i)} \tag{5}$$

in which $I_F(X)$ is the indicator function ($I_F(X) = 1$ when $\widehat{G}(X) \leqslant 0$ and $I_F(X) = 0$ when $\widehat{G}(X) > 0$), $f_x(X)$ is the PDF of the initial multivariate standard Gaussian distribution, $h_x(X)$ is the PDF of the shifted multivariate Gaussian distribution and N_{IS} is the number of samples. Notice that the values of $\widehat{G}(X)$ are calculated using the obtained values of the kriging mean predictors $\mu_{\widehat{G}}$. The accuracy of P_f is measured by its coefficient of variation $COV(P_f)$. This coefficient of variation is calculated as follows:

$$COV(P_f) = \frac{\sqrt{Var(P_f)}}{P_f} \tag{6}$$

where $Var(P_f)$ is the variance of the failure probability estimate. It is calculated by the following equation:

$$Var(P_f) = \frac{1}{N_{IS}-1}\left[\frac{1}{N_{IS}}\sum_{i=1}^{N_{IS}}\left(I_F(X_i)\left(\frac{f_x(X_i)}{h_x(X_i)}\right)^2\right) - P_f^2\right] \tag{7}$$

(3) Identify among the whole population of N_{IS} samples, the 'best' next candidate sample for which one will compute the performance function value using FLAC^{3D}. This is performed by evaluating a learning function U for each sample in the population. The learning function U usually employed in the kriging-based approaches is given by (cf. [18,19]):

$$U(X_i) = \frac{|\mu_{\widehat{G}(X_i)}|}{\sigma_{\widehat{G}(X_i)}},\ i = 1, \ ..., N_{IS} \tag{8}$$

The 'best' next sample is the one that has the smallest U value [i.e. min(U)]. It should be noted here that the 'best' chosen sample is the one that mostly improves the limit state surface (G = 0) of the meta-model because min(U) searches for the sample that has a small kriging predictor (i.e. a sample that is close to the limit state surface) and/or a high kriging variance (i.e. a high uncertainty in the sign of its performance function value).

(4) If the obtained minimum value of U is smaller than 2, evaluate the performance function value based on FLAC^{3D} for this 'best' candidate and update the DoE by adding the new 'best' sample. Also, reconstruct the kriging meta-model again with the updated DoE.

(5) Repeat the steps 2–4 several times until the smallest U value

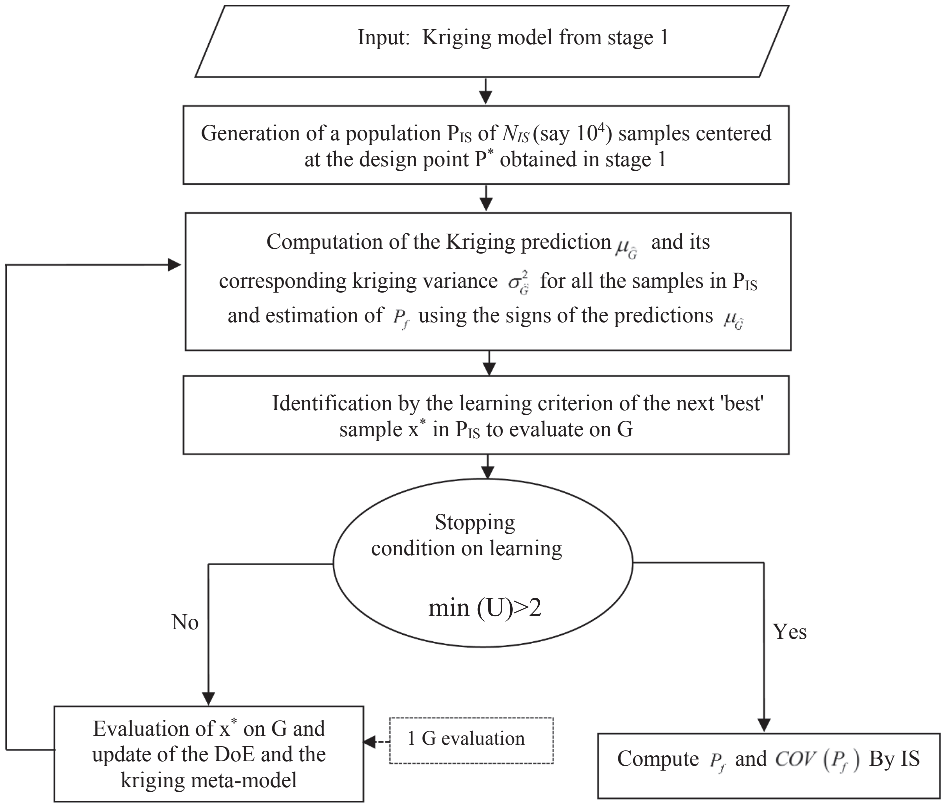

Fig. 2. Flowchart of the proposed AK-IS procedure (Stage 2: Enrichment process).

becomes larger than 2. Notice that the stopping criterion min (U) > 2 corresponds to a maximal probability of making a mistake on the sign of the performance function of $\Phi(-2) = 0.023$ (see [18]).

At this stage, the learning stops and the meta-model is considered sufficiently accurate for the computation of the failure probability. When the learning stops, one must compute the estimated values of both the probability of failure P_f and its corresponding coefficient of variation $COV\,(P_f)$ using the obtained kriging meta-model. It should be emphasized here that the 10,000 evaluations of the learning function U in step 3 were performed for each added sample since the meta-model is continuously changing during the enrichment process. This number of evaluations is much smaller than that used in AK-MCS (i.e. 500,000), thus leading to a much smaller computation time (for a typical added sample) when using AK-IS instead of AK-MCS.

3.3. Numerical implementation

The comprehensive step-by-step procedure described above was implemented in Matlab software. It includes the random field discretization by EOLE method, the determination of the design point by an iterative procedure and the construction of a kriging meta-model for the computation of the failure probability. The implemented Matlab procedure makes several calls to the FLAC^{3D} code for the computation of the system response (i.e. ultimate bearing capacity on a spatially varying soil) or the corresponding performance function value for the different soil realizations. The computation of the system response *via* FLAC^{3D} software was not presented herein to avoid repetition and the reader may refer to [12,13] for more details.

4. Probabilistic numerical results

Before the presentation of the probabilistic results of a spatially varying soil, it seems necessary to validate the present AK-IS procedure by comparison of its results with those obtained by [19] when considering a simple analytical equation. This is the aim of the next subsection.

4.1. Validation of the present AK-IS procedure via a simple analytical equation

This section focuses on the validation of the present AK-IS procedure through an analytical example. The corresponding performance function is given as follows:

$$G(u_1, u_2) = 0.5(u_1 - u_2)^2 - 1.5(u_2 - 5)^3 - 3 \tag{9}$$

where u_1 and u_2 are two standard normal random variables. A comparison between the results obtained by the present AK-IS procedure and those provided by [19] was presented in Table 1. Notice that in [19], the design point was determined using the classical FORM analysis based on the analytical equation of the performance function.

Table 1
Probabilistic outputs and the corresponding number of calls to the performance function N_{calls} as obtained from the two AK-IS methods.

Method	N_{calls}	$P_f \times 10^{-5}$	$COV\,(P_f)$ (%)	β_{HL}	Design point (u_1, u_2)
AK-IS by [19]	19 (DoE) + 7 (enrichment) = 26 samples	2.86	2.39	3.93	(0.788, 3.853)
Present AK-IS approach	15 + (2 × 5) + 4 = 29 samples	2.83	2.40	3.93	(0.786, 3.853)

However; in the present AK-IS procedure, this design point is determined by employing the iterative procedure proposed in the previous section. The aim is to check and validate the proposed iterative procedure which will be employed hereafter in the complex case of the spatially varying soil properties.

As may be seen from Table 1, the approximate kriging meta-model (which was needed for the determination of the design point) was constructed using an initial design of experiments of 15 samples and five iterations with 2 samples per iteration. The enrichment process required 4 additional samples. Thus, the total number of samples (or the number of calls to the performance function) needed in our procedure is equal to 29 samples. This number is close to that needed by the classical FORM analysis by [19] (i.e. 26 samples) with the advantage that the present approach may be applied to analytically-unknown performance functions.

As a conclusion, the iterative procedure proposed in this paper for the computation of the design point can be considered as a powerful tool and may be used for more complex cases involving spatially varying soil properties.

4.2. Probabilistic results in the case of a spatially varying soil

This section aims at presenting the impact of the soil spatial variability on the failure probability against soil punching of a strip footing subjected to a vertical loading. The soil cohesion c and angle of internal friction φ were modeled as two non-isotropic non-Gaussian random fields. The EOLE methodology was used to discretize the two random fields. The illustrative statistical parameters of these two random fields are presented in Table 2. Recall here that the same autocorrelation function (square exponential) was used for both c and φ. Notice also that the soil dilation angle ψ was considered to be related to the soil angle of internal friction φ by $\psi = 2\varphi/3$. This means that the soil dilation angle was implicitly assumed as a random field that is perfectly correlated to the soil angle of internal friction random field.

The performance function employed in the analysis is given by the following equation:

$$G = \frac{q_u}{q_s} - 1 \tag{10}$$

where q_u is the ultimate bearing capacity computed using FLAC^{3D} model making use of the generated realizations of c and φ, and q_s is the footing applied loading. Concerning the mechanical model, a strip footing of breadth B = 1 m that rests on a soil domain of width 13B and depth 5B was considered in the analysis. As mentioned above, this mechanical model was not provided herein and the reader may refer to [12,13].

Finally, notice that the number N_{IS} of samples used in most subsequent configurations was equal to 10,000 samples. This number was found to provide (for these configurations) a small value of the coefficient of variation on the failure probability (< 5%) as it will be shown later. The small size of the sampling population may be explained by the fact that the sampling is performed according to a probability distribution that is centered at the design point leading to a much larger number of samples lying in the failure domain as compared to AK-MCS methodology.

Table 2
Illustrative statistical parameters of the uncertain soil properties.

Random fields	Mean value (μ)	Coefficient of variation COV (%)	Probability density function (PDF)
Soil cohesion (c)	20 kPa	25	Lognormal
Soil friction angle (φ)	30°	10	Beta

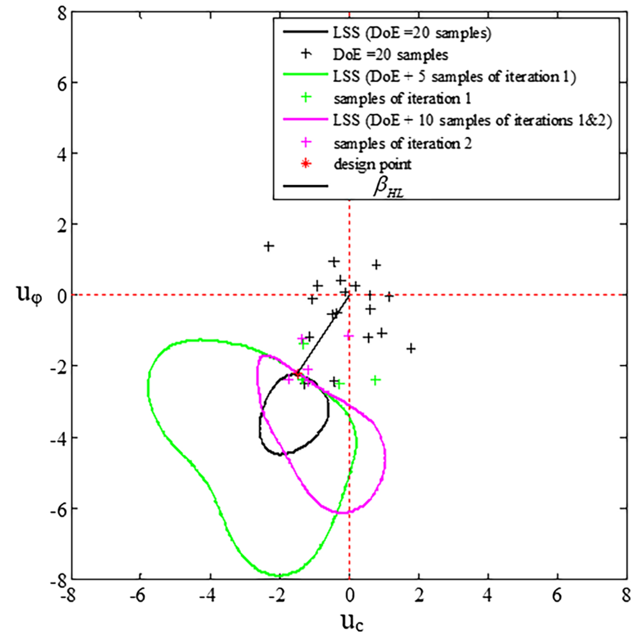

Fig. 3. Evolution of the limit state surface with the addition of new samples during the different iterations of stage 1 when $a_x = a_y = 10{,}000$ m.

4.2.1. Evolution of the limit state surface during the computational process

As was previously mentioned in this paper, the AK-IS procedure consists of two main stages: The first stage (called stage 1) consists in computing the design point from an approximate kriging meta-model constructed using a small number of samples. In the second stage (called stage 2), the approximate meta-model is successively improved through an enrichment process. In this section, the evolution of the limit state surface with the addition of new samples (or realizations) during the two stages (i.e. stage 1 and stage 2) was investigated (see Figs. 3 and 4). A typical case where $a_x = 10{,}000$ m and $a_y = 10{,}000$ m was considered in these figures. This configuration was chosen because it requires only two random variables and thus, the limit state surface can be easily visualized since only a two-dimensional space is needed in

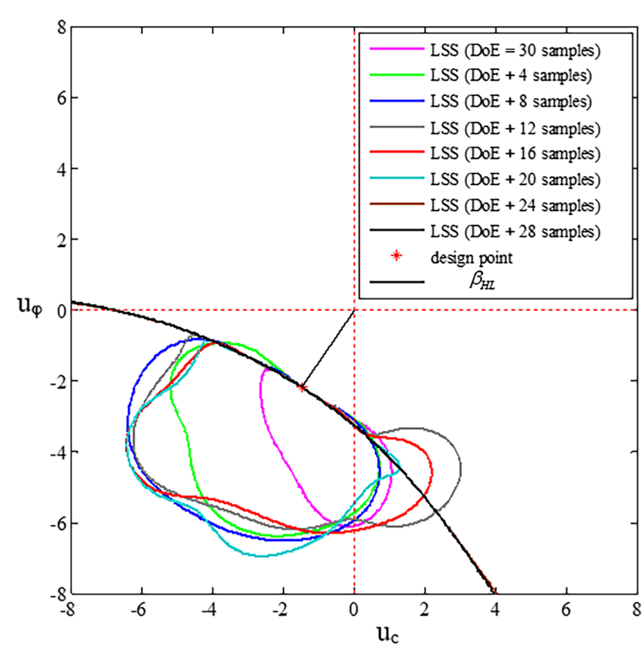

Fig. 4. Effect of the number of added samples during the enrichment process on the limit state surface when $a_x = a_y = 10{,}000$ m.

Table 3
The evolution of the reliability index for the different iterations.

Case	β_{HL}
Initial DoE = 20 samples	2.6205
Initial DoE + 5 samples of iteration 1 = 25 samples	2.6345
Initial DoE + 5 samples of iteration 1 + 5 samples of iteration 2 = 30 samples	2.6340

this case.

Fig. 3 presents the evolution of the limit state surface with the addition of new samples during the different iterations of stage 1. Also, Table 3 presents the evolution of the reliability index β_{HL} for the different iterations. This table shows that the accurate value of the reliability index was obtained from the first iteration in the present case of a homogeneous soil where $a_x = a_y = 10{,}000$ m. Notice however that a larger number of iterations was found necessary (N_2 is between 3 and 13) for spatially varying soil mediums as may be seen from the sixth column of Table 5. It should be remembered here that only the point of the limit state surface which is the closest one to the origin of the standard coordinates system is expected to be correct at the end of the first stage of AK-IS; the other points of this limit state being in general not correctly estimated within this stage.

Fig. 4 presents the evolution of the limit state surface with the addition of new samples (from zero to 28 samples) during stage 2; the number 28 being the needed number of added realizations during the enrichment process. From this figure, one may notice that the limit state surface is successively improved with the addition of new samples. Notice however that for the last two iterations, the two curves representing the limit state surface are coinciding. Thus, the limit state surface cannot be further improved beyond 24 samples. This means that there is no bias in the meta-model beyond 24 samples.

Table 4 presents the evolution of the probability of failure P_f and its corresponding coefficient of variation $COV(P_f)$ as function of the added samples.

This Table shows that the values of P_f and $COV(P_f)$ converge after the addition of 24 samples. This is in conformity with Fig. 4 in which no further improvement in the limit state function was obtained between the last two iterations.

4.2.2. Evolution of the probabilistic outputs during the enrichment process

First of all, recall here that the failure probability is computed each time a new sample is added during the enrichment process. Fig. 5 shows the effect of the number of added samples in the enrichment process on P_f and $COV(P_f)$ values for a typical case where $a_x = 10$ m and $a_y = 1$ m. This figure also provides the learning function values for the different added samples. The configuration ($a_x = 10$ m and $a_y = 1$ m) was studied because it represents a practical case requiring a significant number of random variables (32 random variables in the present case as it may be seen from Table 5).

Fig. 5 shows that both P_f and $COV(P_f)$ vary for the small number of added samples. This is due to the inaccuracy of the kriging meta-model when only a small number of realizations were considered. Notice however that both P_f and $COV(P_f)$ tend to converge to a constant value as the number of added samples increases. It should be mentioned here that 921 samples were needed in the enrichment process in addition to the DoE before the algorithm stops (i.e. [min(U)] > 2). The final obtained values of P_f and $COV(P_f)$ are respectively 1.628×10^{-3} and 2.99%.

Table 4
The evolution of the probability of failure and its corresponding coefficient of variation as a function of the added samples during the enrichment process.

Number of added samples	$P_f \times 10^{-3}$	$COV(P_f)$%
0	3.606	1.957
4	4.589	1.736
8	4.171	1.763
12	4.111	1.878
16	3.919	1.772
20	3.828	1.772
24	3.830	1.771
28	3.830	1.771

As may be seen from Fig. 5, the values of P_f and $COV(P_f)$ reach an asymptote when the number of added samples is equal to 823. An additional increase in the number of added samples does not lead to a significant change in the values of P_f and $COV(P_f)$. This means that when the number of added samples becomes equal to 823, the kriging meta-model is accurate enough (i.e. with no bias) and it can be used to calculate a rigorous value of the failure probability.

Fig. 6a and b present two typical non-critical realizations of the soil shear strength parameters corresponding to the safe (G > 0) and failure (G < 0) domains respectively for the adopted reference case (i.e. when $a_x = 10$ m and $a_y = 1$ m). On the other hand, Fig. 6c presents the critical realizations of the soil shear strength parameters corresponding to the obtained design point for the same configuration.

Contrary to Fig. 6a and b, Fig. 6c exhibits a symmetrical distribution of the soil shear strength parameters with respect to the central vertical axis of the foundation. The weaker soil zone is concentrated around the foundation while the stronger soil is far from the foundation. The weak soil zone under the foundation allows the failure mechanism to easily develop through this zone thus reflecting the most prone soil to punching. Concerning the non-critical realizations (corresponding to G > 0 or G < 0), it can be observed that the realizations corresponding to the safe domain exhibits high values of the shear strength parameters (cf. Fig. 6a). The high shear strength parameters resist soil punching and lead to footing safety. On the contrary, smaller values of shear strength parameters were encountered in the soil mass when dealing with the realizations corresponding to the failure domain (cf. Fig. 6b). This allows the failure mechanism to easily develop in the soil leading to soil failure.

In order to better visualize and interpret the distribution of the soil shear strength in the soil mass, Fig. 7 presents the distribution of the soil cohesion and friction angle along a vertical section (taken at the center of the footing) for the realizations presented in Fig. 6. As may be seen from Fig. 7, the non-critical realizations show more fluctuations than the critical realization corresponding to the design point, with large values in the safe realization and small values in the realization corresponding to failure. The distribution of the shear strength parameters corresponding to the critical realization was shown to present fluctuations in the upper part of the soil profile near the foundation (i.e. in the depth affected by the soil failure mechanism) and tends to be nearly uniform in the lower part of the soil. One may also see that smaller values of the soil shear strength parameters were found in the upper part of the soil mass for this critical realization thus allowing the failure mechanism to easily develop within this zone. Higher values of the soil shear strength parameters were observed in the lower part of the soil mass far from the foundation, this zone having negligible influence on the bearing capacity of the foundation.

4.2.3. Parametric study

This section aims at presenting the effect of the autocorrelation distances of the random fields on the probabilistic outputs (i.e. the failure probability and the reliability index).

Fig. 8 presents the effect of the isotropic autocorrelation distance ($a_x = a_y$) on P_f and β_{HL} as obtained from AK-MCS and AK-IS methodologies. Also, Figs. 9 and 10 present the effect of the autocorrelation distance (a_y or a_x) on P_f and β_{HL} as obtained from the same two methodologies. Remember here that the AK-MCS results are those provided by Al-Bittar et al. [23]. However, the AK-IS results are those obtained in the present paper. From Figs. 8–10, one may observe that the two methods lead to similar results. The maximal percent difference

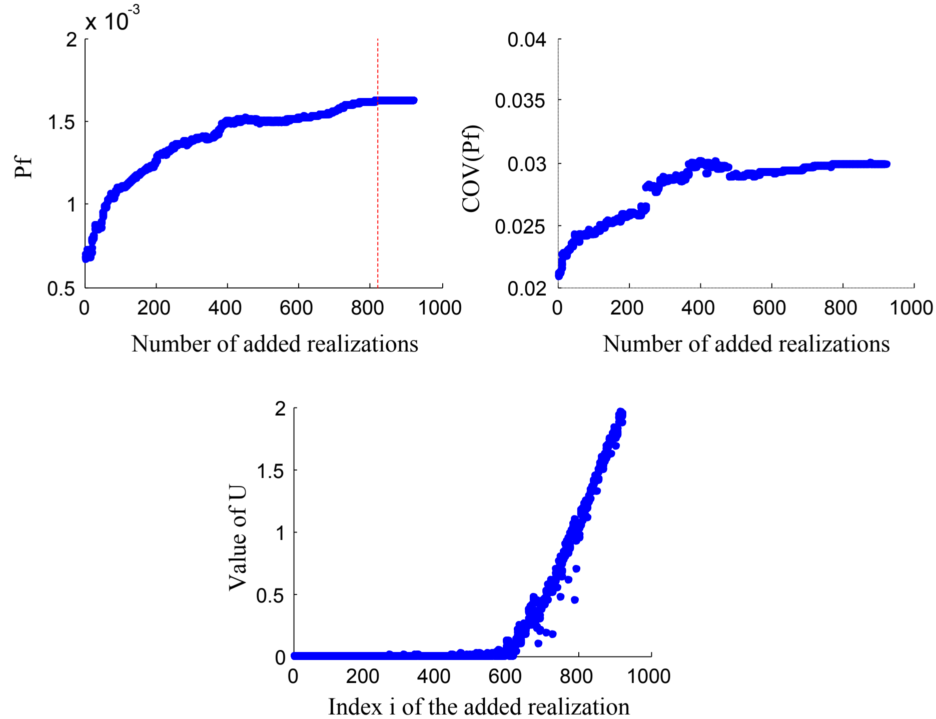

Fig. 5. AK-IS results for a spatially varying soil (a_x = 10 m, a_y = 1 m).

Table 5
Adopted number of random variables and the corresponding value of the variance of error of EOLE together with the values of P_f, $COV\,(P_f)$, size of DoE, number of added realizations and number of calls to the mechanical model for various soil variabilities.

(a) Case of an isotropic case (a_x = a_y)

a_x = a_y (m)	Number of random variables	Variance of the error %	$P_f \times 10^{-3}$	$COV\,(P_f)$%	Size of DoE = $N_1 + 5 \times N_2$	Number of added realizations	Number of calls to the mechanical model
2	62	4.850	0.710	4.64	62 + 5 × 9	2128	2235
3	32	4.647	1.718	5.38	20 + 5 × 6	1076	1126
5	24	0.953	2.738	2.42	20 + 5 × 10	812	882
10	10	0.815	3.404	1.91	20 + 5 × 10	243	313
20	8	0.170	3.745	1.91	20 + 5 × 3	200	235
50	6	0.016	3.831	1.82	20 + 5 × 6	74	124
100	6	0.001	3.933	1.82	20 + 5 × 6	90	140

(b) Case of an anisotropic case (a_x=10 m with varying a_y)

a_y (m)	Number of random variables	Variance of the error %	$P_f \times 10^{-3}$	$COV\,(P_f)$%	Size of DoE = $N_1 + 5 \times N_2$	Number of added realizations	Number of calls to the mechanical model
0.5	60	4.619	0.313	2.898	60 + 5 × 8	1937	2037
0.8	38	4.798	1.234	3.51	20 + 5 × 8	1192	1252
1	32	4.212	1.628	2.99	20 + 5 × 8	921	981
2	24	1.437	2.755	2.68	20 + 5 × 5	644	689
5	12	1.682	3.172	2.06	20 + 5 × 8	354	414
10	10	0.815	3.404	1.91	20 + 5 × 10	243	313
20	8	0.855	3.425	1.98	20 + 5 × 5	228	273
50	8	0.297	3.434	1.99	20 + 5 × 4	210	250
100	8	0.099	3.595	1.78	20 + 5 × 10	194	264

(c). Case of an anisotropic case (a_y=2 m with varying a_x)

a_x (m)	Number of random variables	Variance of the error %	$P_f \times 10^{-3}$	$COV\,(P_f)$%	Size of DoE = $N_1 + 5 \times N_2$	Number of added realizations	Number of calls to the mechanical model
2	62	4.850	0.710	4.64	62 + 5 × 9	2128	2235
5	30	4.101	2.221	2.65	20 + 5 × 13	988	1073
10	24	1.437	2.755	2.68	20 + 5 × 5	644	689
20	16	1.415	3.023	2.07	20 + 5 × 9	437	502
50	12	1.272	3.180	1.87	20 + 5 × 12	313	393
100	10	0.842	3.191	1.82	20 + 5 × 12	244	324

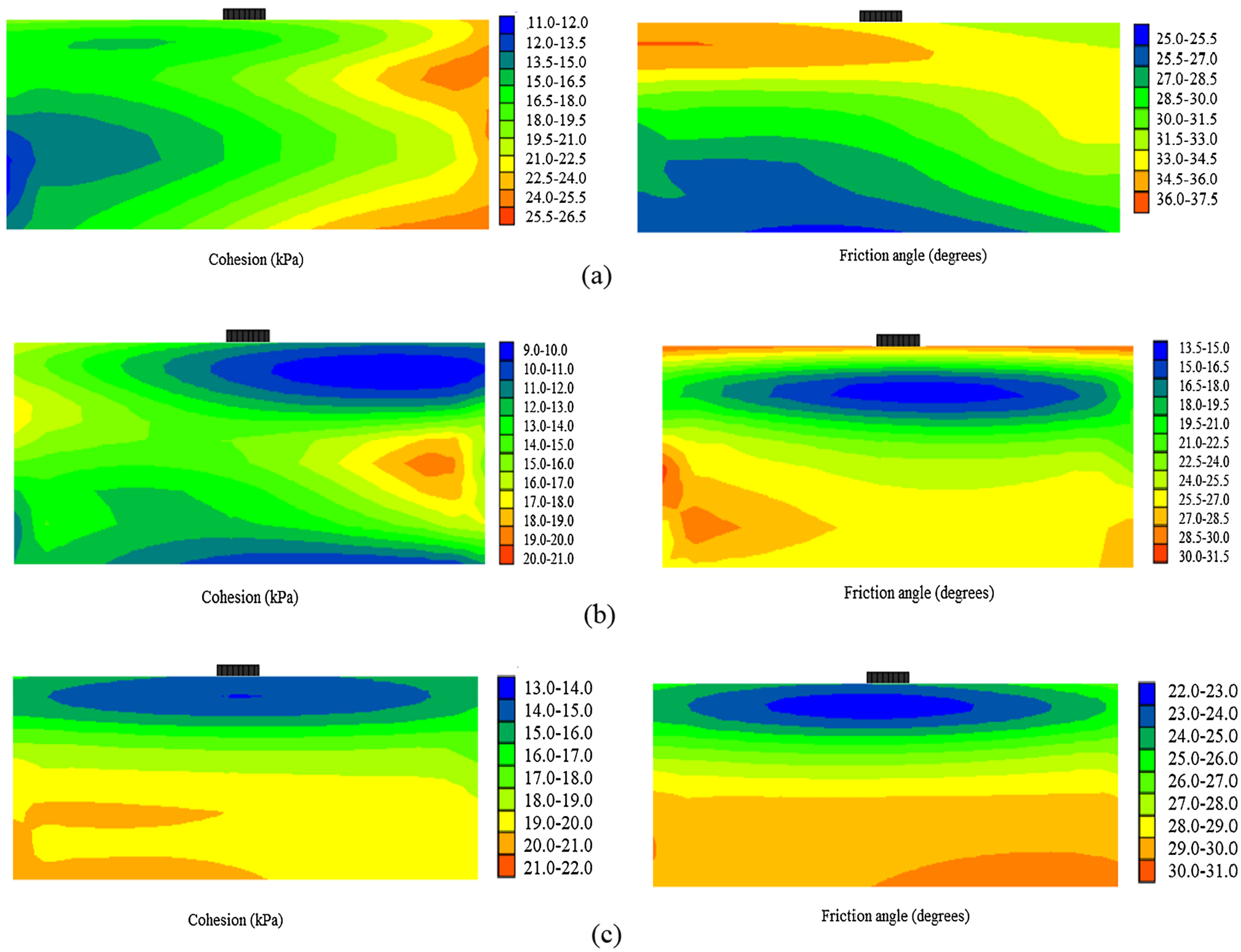

Fig. 6. Typical realizations (a) safe domain, (b) failure domain and (c) design point.

between the two approaches is smaller than 7%.

The values of the probabilistic outputs obtained by AK-IS approach and corresponding to the different soil variabilities were given in Table 5. Columns 2 and 3 of Table 5 provide the number of random variables (or the number of eigenmodes) and the corresponding variance of the error of EOLE methodology for different values of the autocorrelation distances. Columns 4, 5, 6, 7 and 8 of the same table provide the failure probabilities, the corresponding values of the coefficient of variation, the size of the DoE (where DoE = $N_1 + 5 \times N_2$), the number of added realizations and the total number of calls to the

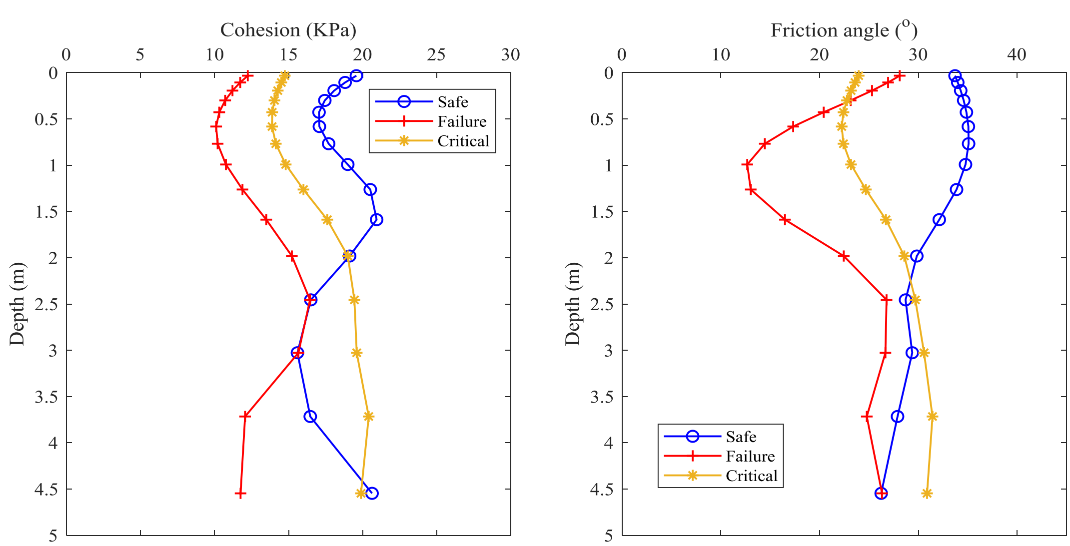

Fig. 7. Vertical cross-section at the footing center.

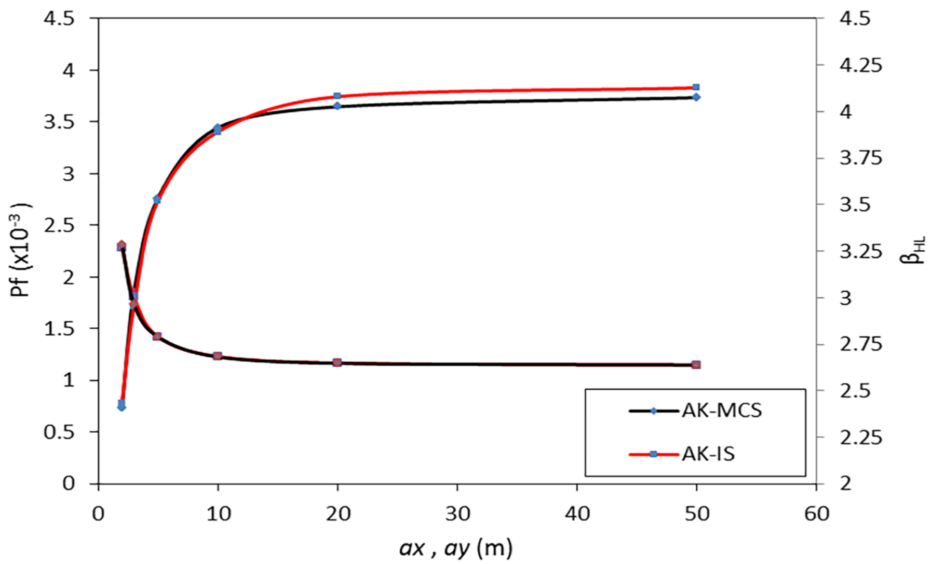

Fig. 8. Effect of the isotropic autocorrelation distance $a_x = a_y$ on P_f and β_{HL}.

mechanical model [i.e. DoE+ Number of added realizations]. Remember here that a maximal value of 5% was adopted in this paper for the variance of the error of EOLE methodology. As may be seen from Table 5, the required number of random variables is small for the very large values of the autocorrelation distances and significantly increases for the small values of the autocorrelation distances.

Table 5 shows that the number $N_1 = 20$ samples suggested in the flowchart of Fig. 1 was sufficient for moderate to large values of the autocorrelation distances; however, a higher number of samples was found necessary when dealing with the two configurations corresponding to the small values of the autocorrelation distances [i.e. ($a_x = a_y = 2$ m) and ($a_x = 10$ m, $a_y = 0.5$ m)] for which a large number of random variables (about 60 random variables) was required. The greater number of samples needed for the configurations corresponding to the small values of the autocorrelation distances may be explained by the likely increasing non-linearity of the limit state surface for these cases of very heterogeneous soils. It was found that adopting a value of N_1 that is equal to the number of eigenmodes is a suitable choice (to be able to obtain a first tentative design point) for these configurations.

Table 5 also shows that the number of samples generated around the successive tentative design points was equal to 5 as suggested in the flowchart of Fig. 1. This small number of samples was found sufficient even for the small values of the autocorrelation distances. This may be explained by the fact that the initial construction of the approximate metamodel (that is used to determine the first tentative design point) is the most difficult task when dealing with the determination of the design point. Once an initial tentative design point was detected, the determination of the subsequent tentative design points becomes quite straightforward. Furthermore, the number of iterations that is needed to reach the final design point is quite small (between 3 and 13).

Concerning the IS sampling population, the adopted number N_{IS} of samples determines the coefficient of variation of the computed failure probability. The required number N_{IS} of samples was determined in this paper for the two following configurations [i.e. ($a_x = a_y = 3$ m) and ($a_x = 10$ m, $a_y = 0.8$ m)] corresponding to moderate values of the autocorrelation distances. This number was based on a small target value of the coefficient of variation on P_f of about 5%. It was found equal to about 10,000 samples. The number $N_{IS} = 10{,}000$ samples was then adopted for all the other configurations corresponding to larger values of the autocorrelation distances (where larger values of the failure probability are expected). The obtained values of the coefficient of variation for these configurations were smaller than 5%. This is because

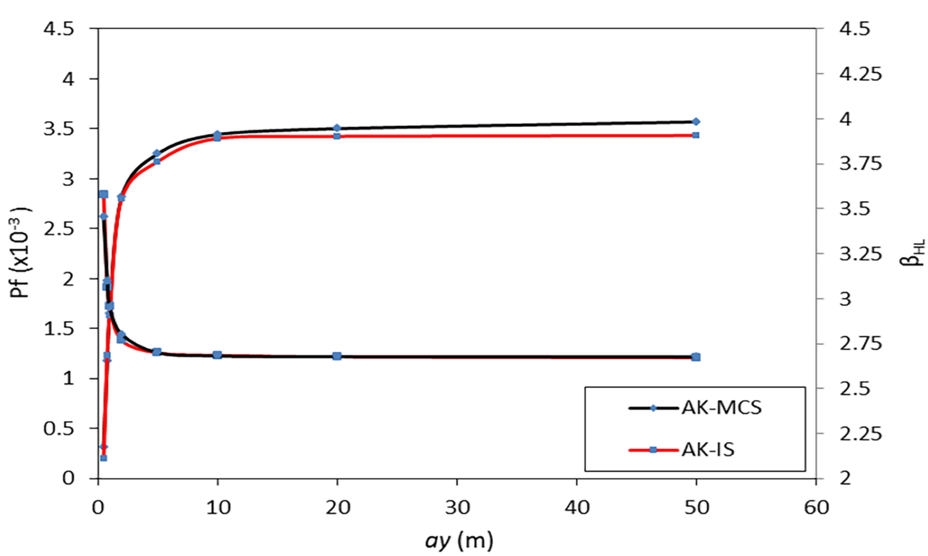

Fig. 9. Effect of the vertical autocorrelation distance a_y on P_f and β_{HL} when $a_x = 10$ m.

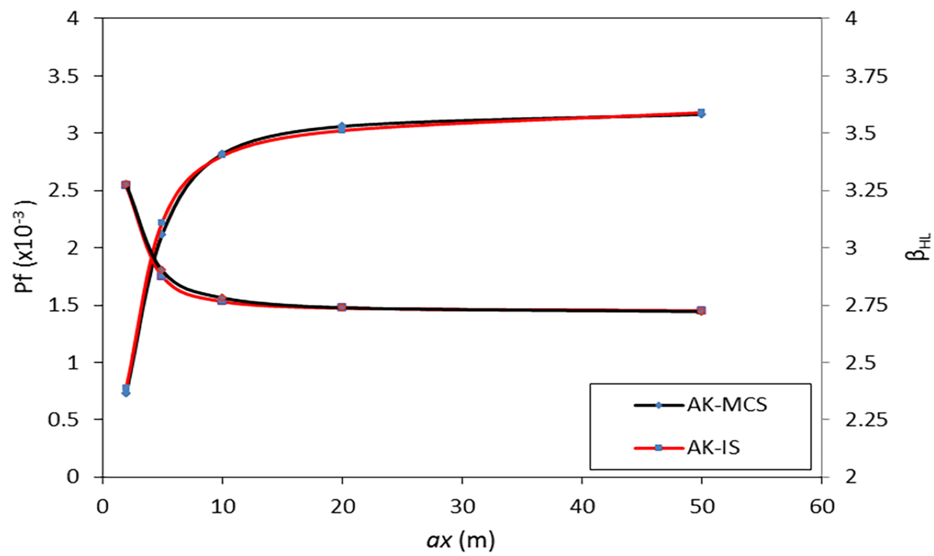

Fig. 10. Effect of the horizontal autocorrelation distance a_x on P_f and β_{HL} when $a_y = 2$ m.

for a prescribed number N_{IS} of samples, the coefficient of variation on the failure probability is smaller for the larger values of the failure probability. Finally, notice that the two configurations [i.e. ($a_x = a_y = 2$ m) and ($a_x = 10$ m, $a_y = 0.5$ m)] corresponding to small values of the autocorrelation distances and thus to quite small values of the failure probability have led to large values of the coefficient of variation when adopting $N_{IS} = 10{,}000$ samples. Thus, an increase in the number N_{IS} of samples was needed for these cases. A number N_{IS} of 30,000 samples was adopted for these configurations. The resulting values of the coefficient of variation on the failure probability were found to be smaller than 5%. The small values of the coefficient of variation obtained in this paper (smaller than about 5% for all configurations) indicate that accurate results were obtained.

Finally, one may observe from Table 5 that the number of added realizations (and the corresponding total number of calls to the mechanical model) required to lead to a good approximation of the kriging model seems to be larger for the smaller values of the autocorrelation distance (because of the likely increasing nonlinearity of the meta-model in the case of highly heterogeneous soils), although there is no regular increase in the number of added realizations with the decrease in the autocorrelation distance. Indeed, this number depends on the evolution of the kriging meta-model during the enrichment process.

4.2.4. Comparison with other probabilistic approaches

In order to compare the P_f value obtained by the present AK-IS approach to that computed by the crude MCS methodology, the reader may refer to the crude MCS results provided in Al-Bittar et al. [23] for the reference case $a_x = 10$ m and $a_y = 1$ m. These MCS results were not detailed herein in order to avoid repetition. Notice that 136,959 calls to the mechanical model were performed while running the crude MCS methodology.

The values of P_f and $COV(P_f)$ obtained from the crude MCS are respectively 1.701×10^{-3} and 6.54%. These values are to be compared with the present AK-IS results [i.e. $P_f = 1.628 \times 10^{-3}$ and $COV(P_f) = 2.99\%$] and the results obtained by Al-Bittar et al. [23] using AK-MCS approach [i.e. $P_f = 1.656 \times 10^{-3}$ and $COV(P_f) = 3.47\%$]. The results provided by the three approaches show good agreement in term of the value of P_f. As a conclusion, the present AK-IS approach gives a quasi-similar value of P_f as the crude MCS method (considered as a reference methodology for the probabilistic analysis). Furthermore, AK-IS is more efficient than AK-MCS because of the smaller sampling population adopted in this approach as compared to AK-MCS method. This leads to a significant reduction in the computation time during the enrichment process as it will be explained below.

The time required by the meta-model (for each added sample) to perform 500,000 evaluations of the learning function U in AK-MCS (where the learning function is based on the mean prediction and the variance prediction by the meta-model) is more significant than that required to perform 10,000 evaluations of this learning function in AK-IS as it was mentioned before. Furthermore, the number of added samples (and the resulting total number of calls to the mechanical model) in AK-IS is either greater or smaller than that needed in AK-MCS (see Table 6) but it remains in the same order as the number of added samples needed in AK-MCS except for the configurations corresponding to the very small values of the failure probability (because of the greater number of samples close to the limit state surface in the case of AK-IS approach). As a conclusion, the computation time required by AK-IS during the enrichment process (which is equal to the number of added samples multiplied by the time required to compute the 10,000 evaluations of the learning function U by the meta-model) is much smaller than that required by AK-MCS that makes use of a quite similar number of added samples with a much greater computation time needed for the 500,000 evaluations of the learning function U by the meta-model. For instance; when considering the typical case where $a_x = 10$ m and $a_y = 2$ m, 12 days (in average) were necessary to complete the AK-MCS computation, whereas only 3 days were needed in average to perform a complete calculation using the AK-IS method.

5. Conclusion

The popular active learning reliability method (called AK-IS) by Echard et al. [19] which is a combination of kriging metamodeling and importance sampling is used in this paper for the probabilistic analysis of geotechnical structures involving spatially varying soil properties. More specifically, the probabilistic model developed in this paper aims at computing the probability of failure against soil punching of a strip footing resting on a spatially varying soil and subjected to a vertical load. The soil cohesion and angle of internal friction were modeled by two non-isotropic non-Gaussian random fields that share an identical square exponential autocorrelation function. The soil cohesion was modelled by a log-normal distribution and the soil angle of internal friction was modeled by a beta distribution. EOLE methodology was used for the discretization of the two random fields.

As is well-known, AK-IS approach has the advantages of both kriging (by using the prediction mean and prediction variance for the

Table 6
Number of added realizations and number of calls to the mechanical model as needed by AK-MCS and AK-IS for various soil variabilities.

(a) Case of an isotropic case ($a_x = a_y$)

$a_x = a_y$ (m)	AK-MCS		AK-IS	
	Number of added realizations	Number of calls to the mechanical model	Number of added realizations	Number of calls to the mechanical model
2	742	762	2128	2235
3	995	1015	1076	1126
5	870	890	812	882
10	286	306	243	313
20	210	230	200	235
50	105	125	74	124
100	100	120	90	140

(b) Case of an anisotropic case (a_x = 10 m with varying a_y)

a_y (m)	AK-MCS		AK-IS	
	Number of added realizations	Number of calls to the mechanical model	Number of added realizations	Number of calls to the mechanical model
0.5	427	447	1937	2037
0.8	790	810	1192	1252
1	752	772	921	981
2	672	692	644	689
5	406	426	354	414
10	286	306	243	313
20	190	210	228	273
50	239	259	210	250
100	232	252	194	264

(c) Case of an anisotropic case (a_y = 2 m with varying a_x)

a_x (m)	AK-MCS		AK-IS	
	Number of added realizations	Number of calls to the mechanical model	Number of added realizations	Number of calls to the mechanical model
2	742	762	2128	2235
5	824	844	988	1073
10	672	692	644	689
20	494	514	437	502
50	357	377	313	393
100	256	276	244	324

determination of the 'best' new candidate sample to be evaluated by the computationally expensive mechanical model) and importance sampling (for the generation of samples around the most probable failure point). Indeed, contrary to the active learning method AK-MCS by Echard et al. [18] combining kriging and Monte Carlo simulations (in which the learning function is computed *via* the meta-model for the whole Monte Carlo population for each added point during the enrichment process), the AK-IS method solves this problem by sampling around the design point using a much smaller size of the sampling population. This significantly reduces the computation time.

This paper presents a simple and non-expensive iterative procedure based on kriging metamodeling for the determination of the design point in the present case of spatially varying soil properties. The other probabilistic procedure related to the enrichment process is quite similar to that of the original AK-IS methodology by Echard et al. [19].

The main findings of this study in terms of the obtained numerical probabilistic results can be summarized as follows:

(1) The present AK-IS procedure was shown to be much more efficient than AK-MCS in the present case of spatially varying soil properties. It provides an accurate value of the failure probability (i.e. with a small value of the coefficient of variation on this failure probability) needing a much smaller computation time as compared to AK-MCS. The reduced computation time results from the fact that the time required by the meta-model (for each added sample) to perform 500,000 evaluations of the learning function in AK-MCS is more significant than that required to perform 10,000 evaluations of this learning function in AK-IS. As a conclusion, AK-IS significantly reduces the computation time compared to AK-MCS. For instance; when considering the typical case where a_x = 10 m and a_y = 2 m, 12 days (in average) were necessary to complete the AK-MCS computation, whereas only 3 days were needed in average to perform a complete calculation using the AK-IS method.
(2) The critical realizations at the design point have shown a symmetrical distribution of the soil shear strength parameters with respect to the central vertical axis of the foundation with a weak soil zone near the footing.

The main findings and the limitation of this study in terms of the developed methodology can be summarized as follows:

(1) The developed procedure related to the determination of the design point was shown to be a powerful tool since it can handle complex problems involving spatially varying soil properties where an analytically-unknown performance function with a quite large number of random variables (of about 60 random variables) may be involved in the analysis.
(2) Similarly to AK-MCS, the AK-IS kriging approach significantly reduces the number of calls to the mechanical model as compared to the variance reduction techniques usually used in the geotechnical literature in the case of spatially varying soils. Also, AK-IS approach significantly reduces the computation time related to the number of the predictions by the meta-model as compared to AK-MCS. Despite these advantages, AK-IS remains insufficient in the case of very

heterogeneous soils [i.e. when $(a_x = a_y) < 2\,\text{m}$] because a large number of calls to the mechanical model (> 2000 calls) is needed for those cases. More advanced probabilistic approaches are desired for these configurations.

References

[1] Andersen LV, Vahdatirad MJ, Sichani MT, Sørensen JD. Natural frequencies of wind turbines on monopile foundations in clayey soils—a probabilistic approach. Comput Geotech 2012;43:1–11.

[2] Vahdatirad MJ, Andersen LV, Ibsen LB, Sørensen JD. Stochastic dynamic stiffness of a surface footing for offshore wind turbines: implementing a subset simulation method to estimate rare events. Soil Dyn Earthq Eng 2014;65:89–101.

[3] Ahmed A, Soubra AH. Probabilistic analysis at the serviceability limit state of two neighboring strip footings resting on spatially varying soil. Struct Saf 2014;49:2–9.

[4] Yuan J, Papaioannou I, Straub D. Reliability analysis of infinite slopes under random rainfall events. In: Schweckendiekeditor. Geotechnical Safety and Risk V IOS Press; 2015. https://doi.org/10.3233/978-1-61499-580-7-439.

[5] Li DQ, Xiao T, Cao ZJ, Phoon KK, Zhou CB. Efficient and consistent reliability analysis of soil slope stability using both limit equilibrium analysis and finite element analysis. Appl Math Model 2016;40:5216–29.

[6] Jiang SH, Huang JS. Efficient slope reliability analysis at low-probability levels in spatially variable soils. Comput Geotech 2016;75:18–27.

[7] Li DQ, Xiao T, Cao ZJ, Zhou CB, Zhang LM. Enhancement of random finite element method in reliability analysis and risk assessment of soil slopes using Subset Simulation. Landslides 2016;13:293–303.

[8] Xiao T, Li DQ, Cao ZJ, Au SK, Phoon KK. Three-dimensional slope reliability and risk assessment using auxiliary random finite element method. Comput Geotech 2016;79:146–58.

[9] Huang J, Fenton G, Griffiths DV, Li DQ, Zhou CB. On the efficient estimation of small failure probability in slopes. Landslides 2017;14:491–8.

[10] Jiang S, Huang J, Zhou C. Efficient system reliability analysis of rock slopes based on Subset simulation. Comput Geotech 2017;82:31–42.

[11] Van Den Eijnden AP, Hicks MA. Efficient subset simulation for evaluating the modes of improbable slope failure. Comput Geotech 2017;88:267–80.

[12] Al-Bittar T, Soubra AH. Bearing capacity of strip footings on spatially random soils using sparse polynomial chaos expansion. Int J Numer Anal Meth Geomech 2013;37:2039–60.

[13] Al-Bittar T, Soubra AH. Efficient sparse polynomial chaos expansion methodology for the probabilistic analysis of computationally-expensive deterministic models. Int J Numer Anal Meth Geomech 2014;38:1211–30.

[14] Al-Bittar T, Soubra AH. Probabilistic analysis of strip footings resting on spatially varying soils and subjected to vertical or inclined loads. J Geotech Geoenviron Eng ASCE 2014;140:1–11.

[15] Al-Bittar T, Soubra AH. Bearing capacity of spatially random rock masses obeying Hoek-Brown failure criterion. Georisk: Assess Manage Risk Eng Syst Geohazards 2016;11:215–29.

[16] Bourinet JM, Deheeger F, Lemaire M. Assessing small failure probabilities by combined subset simulation and support vector machines. Struct Saf 2011;33:343–53.

[17] Dubourg V, Sudret B, Bourinet JM. Reliability-based design optimization using kriging surrogates and subset simulation. Struct Multidiscip Optim 2011;44(5):673–90.

[18] Echard B, Gayton N, Lemaire M. AK-MCS: An active learning reliability method combining kriging and Monte Carlo Simulation. Struct Saf 2011;33:145–54.

[19] Echard B, Gayton N, Lemaire M, Relun N. A combined importance sampling and kriging reliability method for small failure probabilities with time-demanding numerical models. Reliab Eng Syst Saf 2013;111:232–40.

[20] Bourinet JM. Rare-event probability estimation with adaptive support vector regession surrogates. Reliab Eng Syst Saf 2016;150:210–21.

[21] Moustapha M, Sudret B, Bourinet JM, Guillaume B. Quantile based optimization under uncertainties using adaptive kriging surrogate models. Struct Multidiscip Optim 2016;54(6):1403–21.

[22] Huang X, Chen J, Zhu H. Assessing small failure probabilities by AK-SS: An active learning method combining kriging and subset simulation. Struct Saf 2016;59:86–95.

[23] Al-Bittar T, Soubra AH, Thajeel J. Kriging-based reliability analysis of strip footings resting on spatially varying soils. J Geotech Geoenviron Eng ASCE 2018;144.

[24] Li CC, Der kiureghian A. Optimal discretization of random fields. J Eng Mech ASCE 1993;119:1136–54.

[25] Nataf A. Détermination des distributions de probabilités dont les marges sont données. Comptes-rendus de l'Académie des Sciences 1962;225:42–3.

[26] Sacks J, Welch WJ, Mitchell TJ, Wynn HP. Design and analysis of computer experiments. Statist Sci 1989;4:409–23.

[27] Tandjiria V, Teh CI, Low BK. Reliability analysis of laterally loaded piles using response surface methods. Struct Saf 2000;22:335–55.